Re-Reforming the Bostonian System: A Novel Approach to the School Allocation Problem

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This version: **May 18, 2010**

Comments are welcome

Abstract. This paper proposes ways to modify mechanisms, allocating students to schools, to conciliate Pareto efficiency and stability. Taking as a staring point the recent reform promoted by the *Boston School Committee*, we propose two (independent) slight modifications to reach our objective:

(i) By redefining how the students are prioritized ; and

(ii) By allowing students to exchange the places they have been allocated to, in the actual system, as if they had *property rights* on 'their' places.

Keywords: School allocation problem, matching mechanism, stable allocation, Pareto efficient matching.
Journal of Economic Literature Classification Numbers:
C71 (Cooperative Games),
C72 (Non-cooperative Games),
D71 (Social Choice; Clubs; Committees; Associations)

1 Introduction

Yearly each municipality should allocate each new child-student a place in a school. Different cities use to employ different mechanisms to do such a task, but most of them share a common 'primitive' process to deal with the problem. Such a primitive basically consists in defining how to build priority lists that are often used to break ties when parents' interests for the schools coincide.

The usual way to deal with the problem of how students would be prioritized is solved by building a function that associates each student a score for each school. Usually this function depends on two classes of variables. The first one, that lies on the relationship between the student and each school includes, among others, the distance from the student's residence to the school; and the number of siblings, already attending to the school, the student has. For the second kind of variables, that depends on student's characteristics that are not related to any school, some systems employ the household income; the number of student's household size; the fact that the student had some disability; etc.

A further aspect concerning how the students' scores (relative to each school) are computed comes from the 'qualitative' description of some variables. In fact, even though some of the above-mentioned variables are continuous, they are often 'discretized'. For instance, and relative to the *distance to school* variable, the Seattle School Board uses 'attendance area boundaries'. These boundaries define the schools to which students will receive an initial assignment, based on their address. The use of *School Attendance Area Policies* has been adopted by a wide of school assignment programs. The effect of such a discretization is that, for each school, it is very usual to assign the same score to several students. This fact yields to (necessarily) draw on a lottery to fully unambiguously build a list to describe which student is priority for getting a place when it is contested by some students.

Let us remark that the scoring functions, or the priority orderings they induce, are constructed to reduce an efficiency-equity trade-off. First, to reach a (Rawlsian) efficiency, when allocating places to students, the Public Administration tries to reduce the cost in which households incur, due to the children's attendance to the school. Second, and once 'efficient equivalence classes' has been established, a fair lottery is used to break ties inside each such an efficiency category. The fact that all the students have the same (ex-ante) probability of being ranked at any position, inside their class, induces a kind of ex-ante internal equity relative to each efficiency category.

Therefore, we can think that students' priorities are decided following an efficient intra-equitable criterion. Nevertheless, since some agents are ex-ante prioritized, relative to the final ranking, it is hardly to argue that this criterion is fully equitable.

Even though the criteria employed by the schools to prioritize students can be seen as efficiency criteria, the realization of the procedure, to match students and schools' places, could induce inefficient allocations. This is due to the particular procedure employed to allocate the schools' places among the students.

In this sense, we can mainly find two systems: The centralized ones, that focus on stability, and the decentralized ones, focusing on Pareto efficiency. The mechanism employed in the Boston Area, from 2005, can be seen as a representative example for the first one, whereas the Boston mechanism, used in the same area until that date, is a representative example for the second one. The reader can find a description for both mechanisms in the survey by Sönmez and Ünver (2009).

The literature mainly concentrates on three allocation mechanisms. The first one is known as the 'Boston mechanism'. Its description coincides with the 'Now-or-Never mechanism' introduced by Alcalde (1996) to implement the set of stable allocations in undominated Nash equilibria. The main properties that this mechanism exhibits are

(*a*) When agents do not play strategically, i.e. when the preferences that they declare are their true preferences, the allocation it suggests is Pareto efficient; and

(*b*) When agents act strategically,[1] the expected allocations are stable. More than that, any stable allocation can be decentralized by a Nash equilibrium.

[1] The result by Alcalde (1996) is established for undominated strategies. This is because in the framework analyzed in this paper, agents in both sides of the market are allowed to select strategically their actions. Nevertheless, in the problem modeled in the present paper, since schools are not allowed to act strategically, there is no need to assume that agents use undominated strategies to reach the result.

The second mechanism that has been explored is the 'Student-Optimal Stable mechanism', which coincides with the realization of the 'deferred acceptance' algorithm in which students send proposals to the schools. This mechanism, introduced by Gale and Shapley (1962), always selects a stable allocation. Moreover, all the students (weakly) prefer this allocation to any other stable allocation. Furthermore, when this mechanism is employed, students have no interest on playing strategically.

The third mechanism was proposed, under the name of 'Top Trading Cycles' mechanism, by Abdulkadiroğlu and Sönmez (2003). It is inspired in the homonym algorithm used by Shapley and Scarf (1974) to prove the existence of stable allocations in their 'Housing Markets' model. It is remarkable that this mechanism always selects Pareto efficient allocations and that students have no interest on misrepresent their true characteristics. Nevertheless, the way in which it has been described yields to consider it as hardly used in practice. In our opinion it is not easy to convince the students (or their relatives) about the intuition beyond the employ of such an allocation procedure.

Our approach in the present paper is two-fold. We first explore conditions on agents' characteristics that always solve the efficiency-equity trade off. And, secondly, for the case in which agents' characteristics are unrestricted, we present a new way to deal with the problem, by improving the inefficient solution which is obtained if we impose the equity criterion to be fulfilled.

In the present paper, the first way to deal with the school allocation problem is not related to the allocation procedure, but to the way in which the schools' priority lists are built. Following this approach, the general recommendation is to restrict, for each school, the number of students belonging to its *influence area*. Therefore, and following our first approach, what the *Local School Committees* would do is to redesign their school attendance areas.

Provided that, in our model, we do not assume any particular geographical distribution of students and/or schools, we will assume that schools' priority lists need not to be correlated and, thus, each school might exhibit any priority list to be applied to the students. This is why, throughout this paper, by adopting a poetic license, we refer that *each school is free to decide how it prioritizes the students*. Accordingly, we refer the policy of accurately redesigning each school attendance area as *schools limited freedom* to prioritize the students. In particular, and related to that approach, we show that efficiency and equity are compatible if we proceed as follows. Let us consider two classes of priority lists, namely the 'local' and the schools' lists. Then, for a given school, its priority list is used to prioritize (at most) as much students as it can accept. These students will be prioritized over any other student. The remaining students are prioritized by following the *local committee*'s list.

Relative to the second approach, our recommendations are to find a mechanism which is not centralized, neither decentralized, but a combination of both. In such a case, no restriction is imposed to the schools: They are free to prioritize the students as they wished to. Our proposal is to start by providing a 'tentative' allocation fulfilling an equity criterion. Once this 'equity' is agreed by all the students, we allow them to exchange their places to reach efficiency. The final outcome might fail to satisfy the equity criterion, but such a possible 'criticism' might be considered not acceptable, when some liberal principles would prevail.

What this paper proposes, relative to this approach, is to introduce a solution concept, to be called ε-stability.

The main interest of our new solution concept is inspired in the usual way in which *Public Decision-Makers* act. In fact, what it seems to be the main objective for the Public School Administrators is to allocate students optimally. This can be one of the main reasons supporting systems like the (former) mechanism used in the Boston Area. To guarantee an allocation efficiency, each school orders students according some priority list to reduce the cost to be assumed by their relatives. Following this aim, and based on state and/or local laws, students who live in the attendance area of a school must be given priority for that school over students in the school's attendance area; or siblings of students already attending a school must be given priority. (See, for instance Abdulkadiroğlu and Sönmez (2003))

What seems to be controversial is that the Public Administration might acts against an allocation satisfying (ordinal) efficiency because some 'fairness criterion' should prevail. This controversial is even though stronger if we take into account that the so-called fairness criterion is imposed to reach cardinal efficiency.

On another hand, and following Abdulkadiroğlu et al. (2006), some of the recent reforms in the allocation mechanisms lies on avoiding the possibility of students' strategic behavior. In particular, these authors mention:

> "This interaction between sophisticated and unsophisticated players identifies a new rationale for strategy-proof mechanisms based on fairness, and was a critical argument in Boston's decision to change the mechanism."

Therefore, what it seems to be relevant, when designing a mechanism for school allocation problems, is to accurately combine stability, efficiency and strategy-proofness. Let us notice that, related to our first approach, i.e. restricting the possible priority lists that schools can propose, our results point out that the employ of the new mechanism used in the Boston Area always selects stable and efficient allocations. Moreover, following Alcalde and Barberà (1994), this mechanism is *fully* strategy-proof, i.e. not only students have no interest on misrepresenting their preferences, but it should be also expected that schools do not prioritize students in a *wrong* way.[2]

2 Related Literature

The paper by Balinski and Sönmez (1999) can be viewed as the starting point on a large, recent, literature on solutions for the so called *student allocation problem*.[3] Recently, some authors concentrate on studying and proposing some modifications of different systems employed to allocate school places among students.

[2] In the Valencia Community, Spain, the admission process is similar to the Boston mechanism. The schools prioritize students according to a scoring function that is established by law. Nevertheless, the score that a school assigns to a student is evaluated by the school, and it is not monitored by the Local Public Administration. This lack of motorization allows the schools favor some students, contrary to what legislative norms establish. This behavior can be interpreted as a *manipulation* by some schools administrators.

[3] Some authors made a formal distinction between the model by Balinski and Sönmez (1999) and the school allocation problem presented in

Biró (2008) studies the Hungarian system for both, secondary and higher education; Ergin and Sönmez (2006) analyses, from a game-theoretical point of view, the procedure employed in the Boston Area; Feng (2005) studies the systems employed in Beijing and Shanghai for the high school level; Selim and Salem (2009) explores the ϵ-mechanism, employed in Egypt to solve their high school student-placement problem; Ting (2007) analyzes the college admission problem and the student placement problem in the case of Taiwan, ...

Balinski and Sönmez (1999) introduced a new variant into the analysis of matching markets. They formalized the way in which students should be allocated to the different available positions in the education system. What it is important in their model, and imposes a new way to explore how to study the problem, related to the Matching Theory introduced by Gale and Shapley (1962), is that agents in a side of the market (schools, colleges, institutions, etc.) have no preferences on which their mates are. These agents just declare some lists describing students' priorities when the allocation of some place is contested. These priorities are obtained to reach allocation efficiency. For instance, at primary school level, when students can be considered, from an academic point of view, as undistinguished, the only variables that are considered lie on 'familiar circumstances' (related or not to the schools) as walking distance, sibling, per-capita household income, etc.; for higher education, since students' characteristics are differentiated, students' academic skills are used to decide priorities. Note that, from a *Social* perspective, the employ of academic skills as a relevant variable lies on allocative efficiency. This is because the success of students' effort is more likely if they exhibit the appropriate skills to follow some particular studies.

Given theses primitives, Balinski and Sönmez (1999) provided a way to associate a solution concept in this family of problems to the classical notion of stability introduced by Gale and Shapley (1962). This idea is nice, original and has been useful to introduce some important reforms in the admission systems used at the Elementary School in some areas in the US.[4] The key notion introduced by Balinski and Sönmez (1999) is their *fairness criterion*.[5] It implicitly assumes that some students have *property rights* on certain places. Nevertheless, in their model, students are not free to 'exchange their places'.

Our approach in the present paper is to reconsider the notion of solution to be employed. We adapt the idea of exchange-proofness of an allocation, introduced by Alcalde (1995), to the context of School Allocation Problems. The purpose of such a notion is to guarantee efficiency of allocations. Just to introduce it, let us consider that each student has been allocated a place in some school. And let us assume that there is no student *justifiably envying* some other student's place. Under such a consideration, one might well assume that students have *property rights* on

Ergin and Sönmez (2006). Nevertheless, for our purposes, when the 'academic institutions' do not act strategically, and their only role is providing educational services, we can treat both literatures as coincident ones.

[4] See, for instance the papers by Abdulkadiroğlu et al. (2009), relative to the New York City High School Match, or Abdulkadiroğlu et al. (2006) for the Boston Public School Match

[5] This criterion is also called *non justified envy* by Haeringer and Klijn (2009). We consider both expressions as equivalent throughout the paper.

their places so that they are free to 'exchange' their own places if they wished to. What ε-stability imposes is that students will have no interest on exchanging their school places. The way to reach our objective is simple. We start by allocating each student at the best school she can reach, provided that the allocation must be stable. This job can be done by applying the students-propose deferred algorithm designed by Gale and Shapley (1962). Then, let us consider the Walrasian market where agents are the students, commodities are the schools' places; and each agent's initial endowment is her best stable allocation, as previously stated. This market has the formal structure of a 'Housing Market', as modeled by Shapley and Scarf (1974). Therefore, let us allow the students to exchange 'their' places. As a consequence of this interaction an efficient allocation is reached, in which all the students will be, at least, as well as in any stable allocation, and thus ε-stable.

The rest of the paper is organized as follows. Section 3 introduces the basic framework and provides some definitions which are classical in the literature. Section 4 explores conditions on agents' characteristics under which efficient and stable allocations exist. Our conclusions point out a general difficulty to conciliate both concepts. This 'difficulty' advices us that we should shift the focus on how the allocation problem could be solved. The solution could be found in a reconsideration on how the schools give priorities to the students. We propose a general formulation that can be used as a new way to discuss how the system should be reformed. Section 5 proposes, for a framework in which the previous reform is considered as unsatisfactory by the regulator, how to modify the allocation system, provided that colleges are free to exhibit any way to prioritize students. The aim of Section 6 is to propose a procedure selecting, for each School Allocation Problem, a φ-stable allocation. The way in which this mechanism is described points out how the actual Boston system might be *re-reformed*. Conclusions are gathered to Section 7. Finally, all the proofs are relegated to the Appendix.

3 The School Allocation Problem

This section is devoted to introduce some formalisms related to the School Allocation Problem. This family of problems faces two set of non-empty disjoint agents to be called Students and Schools. The set of Students is denoted by $\mathcal{S}$, and has n individuals, i.e. $\mathcal{S} = \{s_1, \ldots, s_i, \ldots, s_n\}$. The set of Schools is denoted by $\mathcal{C}$, and has m elements, i.e. $\mathcal{C} = \{c_1, \ldots, c_j, \ldots, c_m\}$.

Each school has a number of seats (or places), to be distributed among the students, that will be called its capacity. Let $q_{c_j} \geq 1$ denote school c_j's capacity; and let $Q = \left\{q_{c_1}, \ldots, q_{c_j}, \ldots, q_{c_m}\right\}$ the vector summarizing schools' capacities. Schools are also endowed a *priorities linear ordering* over the set of students. Let $\pi_{c_j} \in \mathbb{R}^n$ be the students' ordering for school c_j and Π the $(m \times n)$-matrix summarizing these priorities. Formally, π_{c_j} is described as a n-dimensional vector such that for each $k \in \{1, \ldots, n\}$ there is a unique student s_i for which $\pi_{c_j s_i} = k$; given this description, the j-th row for matrix Π coincides with vector π_{c_j}.

Note that, under our description, no school would consider a student to be inadmissible. Notice that most of the legislative norms impose such a restriction in the way that the schools rank their potential students. Nevertheless, our model might capture the possibility of a student to be inadmissible at a low cost: just by introducing a new variable for each school defining the *priority level* of the last admissible student.

On the other side, each student has linear preferences over the set of schools, so that no student will consider two different schools as equivalent (or indifferent), and no school is neither considered as inadmissible by any student. Let ρ_{s_i} denote the schools' ranking induced by student s_i's preferences,[6] and Φ the $(n \times m)$-matrix summarizing these rankings. Note that our model assumes that each student considers all the school as admissible.[7] Nevertheless, we can also reformulate this model by assuming that each student might consider some schools as unacceptable. The essence of this paper is the same in both frameworks.

Therefore, a School Allocation Problem can be described by listing the elements above: $\mathcal{SAP} = \{\mathcal{S}, \mathcal{C}; \Phi, \Pi, Q\}$. We will say that a School Allocation Problem is non-scarce whenever there is enough places to allocate all the students

$$\sum_{c_j \in \mathcal{C}} q_{c_j} \geq n.$$

Given a School Allocation Problem, $\mathcal{SAP}$, a solution for it is an application μ that matches students and schools' places. Such a correspondence is called a matching. Formally,

Definition 3.1 [**Matching**]
A *matching* for $\mathcal{SAP}$, a School Allocation Problem, is a correspondence μ, applying $\mathcal{S} \cup \mathcal{C}$ into itself, such that:

(a) For each s_i in $\mathcal{S}$, if $\mu(s_i) \neq s_i$, then $\mu(s_i) \in \mathcal{C}$;

(b) For each c_j in $\mathcal{C}$, $\mu(c_j) \subseteq \mathcal{S}$, and $|\mu(c_j)| \leq q_{c_j}$; [8] and

(c) For each s_i in $\mathcal{S}$, and any c_j in $\mathcal{C}$, $\mu(s_i) = c_j$ if, and only if, $s_i \in \mu(c_j)$.

The central solution concept used through the literature is stability, as defined by Balinski and Sönmez (1999). This stability notion coincides with the pair-wise stability introduced by Gale and Shapley (1962). Under our considerations (i.e., each school is acceptable for any student and vice versa), stability is defined as follows.

Definition 3.2 [**Stability**]
A matching for $\mathcal{SAP}$, say μ, is said *stable* if there is no student-school pair (s_i, c_j) such that

(a) $\mu(s_i) = s_i$, or $\rho_{s_i c_j} < \rho_{s_i \mu(s_i)}$; and

(b) $|\mu(c_j)| < q_{c_j}$, or $\pi_{c_j s_i} < \pi_{c_j s_h}$ for some $s_h \in \mu(c_j)$.

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[6] I.e., $\rho_{s_i c_j} = 3$ indicates that student s_i considers that c_j is her third-best school.
[7] Here, we can also invoke legislative regulations establishing that school attendance is compulsory for the children of certain ages.
[8] Throughout this paper $|T|$ will denote the cardinality of set T.

Throughout this paper, we adopt the convention that $\rho_{s_i\mu(s_i)} = m + 1$ whenever $\mu(s_i) = s_i$.

The idea of instability comes basically from the notion of justified envy. (See Haeringer and Klijn (2009)). Let us consider a matching μ, and let us assume that student s_i prefers to study at school c_j rather that developing her educative formation at her actual school $\mu(s_i)$. If s_i has a priority higher than that of some of the actual students attending school c_j, or this school is still having some vacant, she might claim that the allocation process has been *unfair*.

A second notion that has also been analyzed in this framework is that of efficiency. To introduce appropriately this concept, let us remember that the only role for the schools is to provide educational services needed by the students. Therefore the natural notion of efficiency, as proposed by Balinski and Sönmez (1999) for this framework, is Pareto efficiency (from the students' point of view).

Definition 3.3 [**Pareto Efficiency**]
Given a School Allocation Problem, $\mathcal{SAP}$, we say that matching μ is *Pareto efficient* if for any other matching μ' there is a student, say s_i, such that

$$\rho_{s_i\mu(s_i)} < \rho_{s_i\mu'(s_i)}.$$

Note that, for any non-scarce School Allocation Problem, stability and/or efficiency of a matching μ implies that, for each student s_i, $\mu(s_i) \in \mathcal{C}$.

A *matching mechanism* is a regular procedure that associates to each School Allocation Problem a matching for such a problem. A matching mechanism $\mathcal{M}$ is said to be stable if, for any given problem, it always selects a stable matching. Similarly, we say that a matching mechanism is Pareto efficient whenever its outcome is always Pareto efficient, related to its input. It is easy to see that there are stable matching mechanisms. In fact, any of the versions of the deferred-acceptance algorithms proposed by Gale and Shapley (1962) associates a stable matching for the related School Allocation Problem. On the other hand, the now-or-never mechanism introduced by Alcalde (1996) always selects a Pareto efficient matching when the proposals are made by the students.[9]

The first question that we deal with is the possibility of designing matching mechanisms that always select stable and Pareto efficient allocations. As the next result reports, it might be an impossible task to conciliate the 'fairness' notion evolving stability and Pareto efficiency.

Proposition 3.4 There is no matching mechanism selecting a stable and Pareto efficient allocation for each School Allocation Problem.

Proposition 3.4 points out two main questions. The first one is related to the existence of conditions on Φ and Π guaranteeing the existence of matching mechanisms being stable and Pareto efficient. The second question is related to proposing a new solution concept that accurately combines the notions of fairness, reflected by stability, and Pareto efficiency.

[9] The now-or-never mechanism is also known as the Boston mechanism because it was used in the Boston school district.

4 On the existence of stable, Pareto efficient mechanisms

The aim of this section is to explore conditions under which stability and Pareto efficiency were compatible for some matching mechanisms.

Just as a starting point, we would like mention that the set of stable matchings has a lattice structure. Moreover, one of its extremes can be reached by applying the student-proposing deferred acceptance algorithm. Therefore, the unique stable matching mechanism that could eventually be Pareto efficient is the so called *Student-Optimal Stable mechanism*. Thus, the question that we deal with in this section is: There are (non-trivial) conditions on the rankings and/or the priorities matrices under which the students optimal stable matching is always efficient?

To provide a positive answer to the above question, let us consider the following condition on schools' priorities:

Common Priorities Condition We say that the schools' priorities matrix Π satisfies the Common Priorities Condition, *CPC* in short, if, and only if, there is a (fixed) relation of students' priorities, $\pi = (\pi_{s_1}, \ldots, \pi_{s_i}, \ldots, \pi_{s_n})$, such that for each school c_j, and 'preferred set of students' for such a school, $S^j \subseteq \mathcal{S}$, with $\left|S^j\right| \leq q_{s_j}$, its priority relation satisfies:

(a) $\pi_{c_j s_i} \leq q_{c_j}$, for each $s_i \in S^j$; and

(b) $\pi_{c_j s_i} = \left|S^j\right| + \left|\left\{s_h \in \mathcal{S} \setminus S^j \colon \pi_{s_h} \leq \pi_{s_i}\right\}\right|$, for each student $s_i \notin S^j$.

The idea beyond *CPC* is the following. Let us imagine that students are ordered following a common criterion, which is independent from the schools characteristics (i.e., it does not depend on the distance from the students' residence to the school, neither on whether the students have or not any sibling at the school, ...). Then, once such common priorities have been fixed, each school can modify it by selecting its 'ideal' set of students, whose size must be not higher than the number of its available positions, but the rest of ('non-ideal') students must be prioritized according the *common criterion*.

We can now establish the following result.

Proposition 4.1 Let $\mathcal{SAP}$ be a School Allocation Problem. If Π satisfies *CPC*, then it has only one matching being stable and Pareto efficient.

We can define a condition, similar to *CPC*, applying to students' characteristics. This is the essence of *CRC*.

Common Ranking Condition We say that the students' rankings matrix Φ satisfies the Common Ranking Condition, *CRC* in short, if, and only if, it is possible to find a linear order over the set of schools $\mathcal{C}$, described by the ranking $\rho = \left(\rho_{c_1}, \ldots, \rho_{c_j}, \ldots, \rho_{c_m}\right)$, such that for each student, s_i, and school c_j,

$$\rho_{s_i c_j}(s_i) = \begin{cases} 1 + \rho_{c_j} & \text{if } \rho_{c_k} < \rho_{c_j} \\ \rho_{c_j} & \text{if } \rho_{c_k} > \rho_{c_j} \end{cases}$$

where c_k is such that $\rho_{s_i c_k} = 1$.

Just to introduce the idea underlying *CRC*, let us imagine that there is an 'unambiguous' way to order the schools (that can be derived from some quality index that is commonly accepted), but each student has a preferred school due to some 'particular reasons' (for instance due to its proximity to the student's residence address; or because she has some sibling attending to that school; or because her parents studied at that school; etc.) *CRP* says that all the students will share the same preferences except that they can differ on which her 'best school' is.

Unfortunately, as the next example points out, a result similar to Proposition 4.1 can not be reached just by assuming that students' ranking satisfies *CRC*. We need an extra qualification, which is that all the students share the opinion on which her 'best' school is.

Example 4.2 Let us consider a School Allocation Problem involving three students and two schools, having one vacant each. Let us assume the ranking and priorities matrices are

$$\Phi = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}; \text{ and } \qquad \Pi = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Note that this School Allocation Problem satisfies *CRC*. Nevertheless, the unique stable matching is described as μ, where $\mu(s_1) = c_2$; $\mu(s_2) = c_1$; and $\mu(s_3) = s_3$, which is inefficient.

What this example might suggest is that the non existence of a stable and efficient matching is related to the fact that this School Allocation Problem is scarce. Nevertheless, it is not true. Just to show it, let is imagine that there is a new school, c_3 and the new problem, consistent with the previous one, is described by

$$\Phi' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{bmatrix}; \text{ and } \qquad \Pi' = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Note that, for any $q_{c_3} \geq 1$, the unique stable matching is μ', where $\mu'(s_1) = c_2$; $\mu'(s_2) = c_1$; and $\mu'(s_3) = c_3$.

The next result proposes conditions that, when imposed on students' rankings, guarantee the existence of a stable and Pareto efficient matching.

Proposition 4.3 Let $\mathcal{SAP}$ be a School Allocation Problem. If Φ satisfies *CRC*, and all the students exhibit the same ranking, then there is a unique stable and efficient matching.

Let us mention that the *CRC* was proposed by Alcalde and Barberà (1994) as an example for Top Dominance, a condition guaranteeing the existence of strategy-proof stable matching mechanisms for the marriage problem.[10] It is also easy to check that, given the close (formal) relationship between the School Allocation and the Marriage Problems, *CPC* is also related to the fulfillment of Top Dominance by the schools' declared preferences (or priorities).

[10] A marriage problem, as modeled by Gale and Shapley (1962), can be described as a School Allocation Problem in which all the colleges have just a vacant position.

Definition 4.4 [**Students' Top Dominance**]
Let $\mathcal{R}_i$ be a set of possible rankings for student s_i. We say that $\mathcal{R}_i$ satisfies *Top Dominance* if for each c_j and c_k in $\mathcal{C}$, and any two rankings ρ_{s_i} and ρ'_{s_i} in $\mathcal{R}_i$ if $\rho_{s_i c_j} < \rho_{s_i c_k}$ and $\rho'_{s_i c_k} < \rho'_{s_i c_j}$ then

$$\left\{c_h \in \mathcal{C}\colon \rho_{s_i c_h} \leq \rho_{s_i c_j}\right\} \cap \left\{c_h \in \mathcal{C}\colon \rho'_{s_i c_h} \leq \rho'_{s_i c_k}\right\} = \emptyset.$$

Definition 4.5 [**Schools' Top Dominance**]
Let $c_j \in C$ be a school, with admission quota q_{c_j}, and $\mathcal{P}_j$ be a set of possible priorities for such a school. We say that $\mathcal{P}_j$ satisfies *Top Dominance* if there is a set of students $\mathcal{S}^j \subset \mathcal{S}$, with $\left|\mathcal{S}^j\right| \leq q_{c_j} - 1$, such that for each s_i and s_f in $\mathcal{S}$, and any two priorities π_{c_j} and π'_{c_j} in $\mathcal{P}_j$ if $\pi_{c_j s_i} < \pi_{c_j s_f}$ and $\pi'_{c_j s_f} < \pi'_{c_j s_i}$ then

$$\left\{s_h \in \mathcal{S}\colon \pi_{c_j s_h} \leq \pi_{c_j s_i}\right\} \cap \left\{s_h \in \mathcal{S}\colon \pi'_{c_j s_h} \leq \pi'_{c_j s_f}\right\} \subseteq \mathcal{S}^j.$$

Note that the idea of the Schools' Top Dominance is that schools have some freedom on how to order their 'prioritized' students. But such a freedom is not 'full range' in the sense that the student which is prioritized to fulfill the college's quota (i.e. $\pi_{c_j s_i} = q_{c_j}$) determines which the priority for the rest of students is. In particular, let us observe that, when $q_j = 1$ for each school c_j, the Schools' Top Dominance introduced in Definition 4.5 coincides with the Top Dominance as defined by Alcalde and Barberà (1994).

The next example points out a general difficulty to generalize the notions of *CPC* and/or *CRC* having positive results similar to the ones proposed in propositions 4.1 and 4.3.

Example 4.6 Let us consider a School Allocation Problem involving three students and three schools, having one vacant each. Let us suppose that schools priorities are described by matrix, satisfying Schools' Top Dominance,

$$\Pi = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Now, let us assume that all the students agree that school c_2 is their worst option. In such a case, and when imposing Students' Top Dominance, we have the following two possibilities:

(a) All the students exhibit the same ranking, and

(b) Students might have opposite opinions on how to rank schools c_1 and c_3.

Note that if the first case holds Proposition 4.3 applies and thus, there is a unique (assortative) stable matching, which is also Pareto efficient. Nevertheless, if it is not the case, the rankings matrix might be

$$\Phi = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

and the unique stable matching that this problem has is not Pareto efficient.
As a conclusion we can say that when the students exhibit rankings satisfying Top Dominance and there is a common opinion on ranking some school as the best one (or the worst one) we can have any of the following two conclusions:

(i) If we add an *anonymity* condition, as Alcalde and Barberà (1994) proposed, to the Top Dominance idea, all the students will exhibit the same ranking. Therefore, the conclusion is that there is a unique (assortative) stable matching which is also efficient, as established in Proposition 4.3.

(ii) On the contrary, if such an anonymity is not demanded, we cannot guarantee the existence of a stable and Pareto efficient matching.

The main result in this section proposes a way to reformulate how students would be prioritized. Following the arguments provided in the Introduction, students are prioritized following a scoring function that can be seen as a function which is separable on two classes of variables.

The first type of variables lies on the relationship between any student and each school. Following this conception, it is expected that each student obtains different scores from two schools and also that each school assigns different scores to any two students. Relative to the second class of variables, they are not related to any school. Therefore, we can identify the score that each student obtain, from a school, relative to the first scoring function, with its score by that school;[11] whereas the score assigned to each student, that does not depend on their relationship with any school, can be seen as a score assigned by the *Local School Committee*. The above interpretation allows us to define two scoring functions, to be called the *School Scoring Function*, CS, and the *Local Scoring Function*, LS.

The last question that we deal with, related to this *score-based* way to build schools' priority list, lies on how to combine the *School Scoring Function* and the *Local Scoring Function* to define a *Global Scoring Function*. Following the idea beyond the procedure used by the Seattle School Board, let us define, for each school, its *Priority students* as those for which the *School Scoring Function* is the relevant function for deciding their score. Therefore, we can formally describe the *Global Score* for an student as follows.[12]

Definition 4.7 Let $\mathcal{S}$ and $\mathcal{C}$ be the sets of students and colleges respectively. We define the **Global Score Function** induced by the School Scoring Function $CS : \mathcal{S} \times \mathcal{C} \to \mathbb{R}_+$; the Local Scoring Function $LS : \mathcal{S} \to \mathbb{R}_+$; and the Priority

[11] Let us remember that the scores are determined by the *Local School Committees*, and also that schools do not determine their priority lists. Thus, we are *abusing interpretation* when saying that the school determines the score. We hope the reader not to be induced to confusion due to such an abuse.

[12] The reader can think on different ways to describe a *Global Scoring Function* by, for instance, adding the Local and the School Scoring Functions. Nevertheless, what it is relevant to reach our positive results formalized in Theorem 4.9 is not much related to the particular composition but on the fact that it induces schools priorities satisfying *CPC*.

Function $P : \mathcal{C} \rightarrow 2^{\mathcal{S}}$ as the function $GS : \mathcal{S} \times \mathcal{C} \rightarrow \mathbb{R}_+$ assigning each student-school pair the value

$$GS(s_i, c_j) = \begin{cases} LS(s_i) + CS(s_i, c_j) & \text{if } s_i \in P(c_j) \\ \\ LS(s_i) & \text{otherwise} \end{cases}$$

To introduce the next definition, for a *Local Scoring Function*, LS, let M denote the maximum score allocated to a student,

$$M = \max_{s_i \in \mathcal{S}} LS(s_i) \tag{1}$$

Definition 4.8 Let $\mathcal{S}$ and $\mathcal{C}$ be the sets of students and colleges respectively, and let Q be the vector summarizing school's capacities. Given LS, CS, the Local and School Scoring Functions respectively; and P, the Priority Function, we say that GS, the *Global Score Function* that they determine satisfies the *limited freedom condition*, LFC henceforth, if it fulfills:

(i) For each school c_j, and student s_i,

$$CS(s_i, c_j) > M.$$

(ii) For each school c_j,

$$|P(c_j)| \leq q_{c_j}, \text{ and}$$

(iii) For any two students s_i, s_h, $s_i \neq s_h$,

$$LS(s_i) \neq LS(s_h).$$

Note that, under *LFC*, the induced priority ordering for a school, if it chooses the function CS, allows the school to decide which its 'top' students are, to fulfil its available places. Nevertheless, once some student has been declared not to be one of the q_{c_j}-top students, the school's opinion is not taken into account to determine such a student's priority.

A second aspect that we want to remark is that, even though in our model each priority ordering is linear, Definition 4.8 allows that some students, in the q_{c_j} first positions, share the same priority. Note that this fact does not influence the stability of any matching, even though if it considers that all those students are equaled prioritized.

Theorem 4.9 *Let $\mathcal{S}$ and $\mathcal{C}$ be the sets of students and colleges respectively, and let Q be the school's capacities vector. Then for each 'Scoring Function' GS that satisfies LFC, and any $\mathcal{SAP} = \{\mathcal{S}, \mathcal{C}; \Phi, \Pi, Q\}$, where priorities are induced by GS, there exists a unique stable and Pareto efficient matching.*

Note that Theorem 4.9 establishes that *LFC* is a sufficient condition to guarantee the existence of matchings conciliating the notions of stability and Pareto efficiency. Let us remark that the usual way to prioritize students comes from the employ of some scoring rule. These functions are usually expressed as the addition of two scoring functions. The first one just takes into account the particular characteristics of the students, isolated from the schools, whereas the second one lies on each student's characteristics related to the considered school. This additively separable functional form follows the shape used in Definition 4.8.

We would like to point out that it is very hard to find a result establishing necessary conditions conciliating stability and Pareto efficiency. In particular, if each student is free to rank any school as her 'best school', it is always possible to find non-scarce problems having a stable and Pareto efficient matching. Just, consider that students rankings are such that, for each school c_j the number of students for which this school is the first-ranked, i.e. $\rho_{s_i c_j} = 1$, is not greater than q_{c_j}. In such a case, no matter which the matrix Π is, the matching that associates each student her best school is both stable and Pareto efficient. Therefore, if conditions are established on individual characteristics, it is very hard to have in mind 'natural' necessary conditions guaranteeing the existence of a stable and Pareto efficient matching.

5 ε-Stability: A New Solution Concept

In this section we propose a new solution concept for the School Allocation Problem. It tries to reduce the trade-off between equity (in terms of stability) and efficiency. The central idea to reach our objective is just to restrict which statements, made by some student, are considered as 'admissible' to induce instability of an allocation.

Following a large tradition on cooperative games, it is fairly important to be precise when defining which objections (made by a set of agents) are admissible and which are not. This is the essence for the Bargaining Set introduced by Aumann and Mashler (1964), and some solution concepts that appeared following that paper. The idea beyond the Bargaining Set is that any agent is free to object against an allocation. What she should do is to propose an alternative allocation fitting some properties. Then, if an agent formally objects against an allocation, any other agent might object this *new proposal* in the same fashion that previously did the former agent. That is, any other agent might *counter-object*. What φ-stability of an allocation imposes is that

(*i*) *no agent will object against this allocation, or*
(*ii*) *any objection presented by an agent will be counter-objected.*

Our proposal in the present paper, related to the idea beyond stability, is just to consider objections against an allocation that cannot be counter-objected. To illustrate our proposal, let us analyze the next example.

Example 5.1 Let us consider the following School Allocation Problem.
$\mathcal{S} = \{1,2,3\}$; $\mathcal{C} = \{a,b,c\}$; $Q = (1,1,1)$; and the ranking and priorities matrices are

$$\Phi = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix}, \text{ and } \Pi = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Note that matching μ, with $\mu(1) = a$; $\mu(2) = b$; and $\mu(3) = c$ is not stable. This is because student 2 claims that she has priority, related to student 1, for studying at school a. Now, let us propose to student 2 the following deal:

> "If you are able to propose a matching, preferred by you to μ, and no other student would claim that the new proposal is unfair (as you did when μ was proposed), the new matching will be implemented."

The conclusion will be that student 2 will not be able to propose an alternative matching.
Therefore, and adapting the arguments above relative the Bargaining Set, the process that φ-stability will capture can be informally described as follows. Let us consider a matching μ. Then, any student is free to claim that such an allocation is, from her point of view, unfair. Nevertheless, her 'criticism' has to be supported by proposing which the definitive matching should be. But this 'new proposal' is accepted by the society only if no student is able to show, by employing arguments, similar to used by the former, that the new matching is still unfair.

Definition 5.2 [**Fair Objection**]
Let $\mathcal{SAP}$ be a School Allocation Problem, and let μ be a matching for such a problem. A *fair objection* from student $s_i \in \mathcal{S}$ against μ is a pair (s_i, μ') such that

(*a*) $\rho_{s_i \mu'(s_i)} < \rho_{s_i \mu(s_i)}$, and

(*b*) $|\mu'(\mu'(s_i))| < q_{\mu'(\mu'(s_i))}$, or $\pi_{\mu'(s_i) s_i} < \pi_{\mu'(s_i) s_h}$ for some $s_h \in \mu'(\mu'(s_i))$.

Definition 5.3 [**Counter-objection**]
Let (s_i, μ') be a fair objection against matching μ. A counter-objection from student s_h against (s_i, μ') is a pair (s_h, μ'') that constitutes a *fair objection* against matching μ'.

We say that (s_i, μ') is a *justified fair objection* against μ if it cannot be counter-objected

Definition 5.4 [**φ-Stability**]
Let $\mathcal{SAP}$ be a School Allocation Problem. We say that matching μ is φ-stable if any objection against it can be counter-objected.

Therefore, the idea of φ-stability for μ is that when some student might claim that such a matching is unfair, she is unable to propose an alternative solution that no student would consider it a fair matching.

Note that, for any School Allocation Problem, $\mathcal{SAP}$, the set of φ-stable matchings is a superset of the set of stable matchings. Therefore, the next statement follows.

Proposition 5.5 Let $\mathcal{SAP}$ be a School Allocation Problem. Then, it has a φ-stable matching.

What it is also relevant is that, in general, there are School Allocation Problems having φ-stable matchings that are not stable. Notice that matching μ, proposed in Example 5.1, is unstable, but it is φ-stable for the related problem.

The central solution concept that we propose in this section, ε-stability, comes from the confluence of two solution ideas, namely Pareto efficiency and φ-stability.

Definition 5.6 [**ε-Stability**]
Let $\mathcal{SAP}$ be a School Allocation Problem. We say that matching μ is ε-stable if it is Pareto efficient and φ-stable for $\mathcal{SAP}$.

The next question that we deal with is the existence of ε-stable allocations. Even though that the sets of Stable and Pareto efficient matchings might not intersect (Proposition 3.4), when we concentrate on φ-stable allocations, instead on stable ones, such an intersection is always non-empty.

Theorem 5.7 *Let $\mathcal{SAP}$ be a School Allocation Problem. Then, it has a matching μ which is ε-stable.*

6 The Exchanging Places Mechanism and ε-Stability

The aim of this section is to propose an algorithm that always selects a ε-stable matching. Therefore this section can also be seen as a constructive proof for Theorem 5.7.

The mechanism that we will propose can be introduced as a combination of two algorithms. The first one is the classic students-proposing deferred acceptance algorithm. The second one follows the idea reflected in the Gale's Top Trading Cycle, introduced by Shapley and Scarf (1974).

Since the deferred acceptance algorithm is well-known in the literature, we will concentrate on a description of how the second algorithm works. For, we need some additional definitions.

Definition 6.1 Let $\mathcal{SAP}$ be a School Allocation Problem, and μ a matching. For s_i given, let δ_{s_i} a ranking of the set of students.[13] We say that δ_{s_i} *μ-agrees* ρ_{s_i} if, and only if,

(a) $\delta_{s_i s_i} < \delta_{s_i s_h}$, for each $s_h \neq s_i$ such that $\mu(s_h) \notin \mathcal{C}$,

(b) $\delta_{s_i s_i} < \delta_{s_i s_h}$, for each $s_h \neq s_i$ such that $\mu(s_h) = \mu(s_i)$, and

(c) for each two students s_h and s_l, with $\mu(s_h), \mu(s_l) \in \mathcal{C} \setminus \mu(s_i)$,

$$\delta_{s_i s_h} < \delta_{s_i s_l} \text{whenever } \rho_{s_i \mu(s_h)} < \rho_{s_i \mu(s_l)}$$

By extension, if we denote by Σ the matrix whose i-th row is δ_{s_i}, we say that Σ μ-agrees Φ whenever for each student s_i, δ_{s_i} μ-agrees ρ_{s_i}

Definition 6.2 [**Students' Incidence Matrix**]
Let $\mathcal{S}$ be the set of students, and let Σ be a matrix of students' rankings, whose i-th row represents s_i's ranking. For each subset of students $\mathcal{S}' \subseteq \mathcal{S}$ we define its *incidence matrix* as the $(|\mathcal{S}'| \times |\mathcal{S}'|)$-matrix that associates, to each s_i and $s_h \in \mathcal{S}'$, the value

$$I^{\Sigma}_{\mathcal{S}'}(s_i, s_h) = \begin{cases} 1 \text{ if } \delta_{s_i s_h} < \delta_{s_i s_l} \text{ for each } s_l \in \mathcal{S}' \setminus \{s_h\} \\ 0 \text{ otherwise} \end{cases}$$

[13] This is, for each $k \in \{1, \ldots, n\}$ there is one, and only one, student s_h such that $\delta_{s_i s_h} = k$.

Definition 6.3 Let Σ be a matrix of students' rankings, and $\mathcal{S}' \subseteq \mathcal{S}$ be a subset of students. A *Cycle* for the incidence matrix $I^{\Sigma}_{\mathcal{S}'}$ is a (non-empty) ordered set of students in $\mathcal{S}'$, $\{s^1, \ldots, s^i, \ldots, s^t\}$ such that, for each $i \leq t-1$,

$$I^{\Sigma}_{\mathcal{S}'}\left(s^i, s^{i+1}\right) = I^{\Sigma}_{\mathcal{S}'}\left(s^t, s^1\right) = 1.$$

Note that, since for each subset of students $\mathcal{S}'$, each row of its incidence matrix has a unique element whose value is 1, it is easy to see that

(a) $I^{\Sigma}_{\mathcal{S}'}$ has, at least one cycle, and

(b) each student $s_i \in \mathcal{S}'$ is involved in, at most, one cycle.

We are now ready to introduce the workings for the *Top Trading Cycle algorithm* in our framework.

Definition 6.4 [**The μ-Σ-Top Trading Cycle algorithm**]
Let $\mathcal{SAP}$ be a School Allocation Problem, and μ a matching. Let Σ a matrix μ-agreeing the students' ranking matrix Φ. The *μ-Σ-Top Trading Cycle* algorithm works as follows:

(*Step* 1) Let consider the students' incidence matrix $I^{\Sigma}_{\mathcal{S}}$, and let $\mathcal{S}^1$ be the students belonging to a cycle for $I^{\Sigma}_{\mathcal{S}}$. Then associate each student $s_i \in \mathcal{S}^1$ her mate

$$\mu^{TTC}(s_i) = \mu(s_h) \text{ where } s_h \text{ satisfies } I^{\Sigma}_{\mathcal{S}}\left(s^i, s^h\right) = 1.$$

Let $\mathcal{S}_1 = \mathcal{S} \setminus \mathcal{S}^1$. If $\mathcal{S}_1 = \emptyset$ the algorithm ends, and matching μ^{TTC} is implemented. Otherwise, go to Step 2.

(*Step* k) Let consider the students' incidence matrix $I^{\Sigma}_{\mathcal{S}_{k-1}}$, and let $\mathcal{S}^k \subseteq \mathcal{S}_{k-1}$ be the students belonging to a cycle for $I^{\Sigma}_{\mathcal{S}_{k-1}}$. Then associate each student $s_i \in \mathcal{S}^k$ her mate

$$\mu^{TTC}(s_i) = \mu(s_h) \text{ where } s_h \text{ satisfies } I^{\Sigma}_{\mathcal{S}_{k-1}}\left(s^i, s^h\right) = 1.$$

Let $\mathcal{S}_k = \mathcal{S}_{k-1} \setminus \mathcal{S}^k$. If $\mathcal{S}_k = \emptyset$ the algorithm ends, and matching μ^{TTC}, as described throughout steps 1 to k, is implemented. Otherwise, go to Step $k+1$.

The algorithm ends at the step t for which $\mathcal{S}_t = \emptyset$.

Note that, since the set of students is finite and, for each step k, $|\mathcal{S}_{k+1}| < |\mathcal{S}_k|$, this algorithm always ends in a finite number of steps.

Relative to the output of the μ-Σ-TTC algorithm, for any given μ and each matrix Σ, we can guarantee that the following properties are satisfied:

(a) μ^{TTC} *is Pareto efficient;*

(b) *for each student* s_i,

$$\rho^{TTC}_{s_i\mu(s_i)} \leq \rho_{s_i\mu(s_i)}.$$

(*c*) $\mu = \mu^{TTC}$ *if, and only if, the former matching is Pareto efficient; and*

(*d*) *if students are asked to reveal their rankings to implement matching* μ^{TTC}*, they will obtain no advantage from misrepresenting their rankings.*

Note that the above properties can be seen as a conclusion derived from theorems 4 and 5 in Alcalde-Unzu and Molis (2009).

We can now establish the following result.

Theorem 6.5 *Let* $\mathcal{SAP}$ *be a School Allocation Problem, and* μ^{SO} *its student optimal stable matching. Let* Σ^{SO} *be a matrix* μ^{SO}*-agreeing the students' rankings matrix* Φ*. Then the matching* μ^{TTC} *obtained from applying the* μ^{SO}-Σ^{SO}-*TTC algorithm is* ε*-stable.*

Summarizing the process that we have introduced in this section, let us propose a formal description for what we call the *Exchanging Places Mechanism*.

To describe how this procedure operates, let fix the set of students $\mathcal{S}$, and define, for each student s_i, an 'exchanging-priorities' vector ω_{s_i} that will be understood as a rule for prioritizing exchanges.[14] In other words, let us imagine that student s_i is located a place at school c_j, and she would like to exchange her place to some student attending to school c_t. What ω_{s_i} describes is how s_i orders the students having a place at c_t to (sequentially) propose them such an exchange. Given this vector for each student, we summarize this information by defining a 'ranking matrix' Ω.

The 'Exchanging Places Mechanism' operates as follows. Given a School Allocation Problem, $\mathcal{SAP}$, and matrix Ω, let μ^{SO} be the students optimal stable matching for $\mathcal{SAP}$, and Σ^{SO} a matrix μ^{SO}-agreeing students' rankings matrix Φ, which is obtained by preserving the priorities established in Ω. Then, apply the μ-Σ-Top Trading Cycle for $\mu = \mu^{SO}$ and $\Sigma = \Sigma^{SO}$. The result of this procedure is the outcome for the Exchanging Places Mechanism.

Definition 6.6 [**The Exchange Places Mechanism**] We define the Exchange Places Mechanism as the function that associates to each School Allocation Problem, $\mathcal{SAP}$, and $(n \times n)$-ranking matrix, Ω,[15] the matching μ^* which is obtained by applying the μ-Σ-TTC algorithm, where

(*a*) μ is the student optimal stable matching for $\mathcal{SAP}$, and

(*b*) Σ is the matrix μ-agreeing the students' ranking matrix Φ that, for each s_i and any two students s_h and s_l in $\mathcal{S} \setminus \{s_i\}$ such that $\mu(s_h) = \mu(s_l)$, $\Sigma_{ih} < \Sigma_{il}$ if, and only if $\Omega_{ih} < \Omega_{il}$.

To conclude this section, let us propose the following example to show how the *Exchange Places Mechanism* works.

[14] ω_{s_i} can be determined by the *local school committee* when establishing a students ordering to be employed for breaking ties in schools' scores. For instance, and in order to make up for the draw effect, each student's vector ω_{s_i} might represent the ordering reversing the draw result.

[15] By 'ranking matrix' we mean that it satisfies that for each row i, and any two different columns j, and h, $\Omega_{ij} \neq \Omega_{ih}$

Example 6.7 Let us consider the following Schools Allocation Problem.

$\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\mathcal{C} = \{a, b, c, d\}$, the capacity for each school is 2; and the Rankings and Priorities matrices are

$$\Phi = \begin{bmatrix} 2\,1\,3\,4 \\ 2\,4\,1\,3 \\ 3\,2\,1\,4 \\ 4\,2\,3\,1 \\ 1\,4\,2\,3 \\ 1\,2\,3\,4 \\ 1\,2\,4\,3 \\ 2\,1\,3\,4 \end{bmatrix} ; \text{ and } \qquad \Pi = \begin{bmatrix} 2\,3\,8\,1\,7\,4\,6\,5 \\ 6\,2\,1\,5\,8\,4\,3\,7 \\ 7\,5\,6\,8\,2\,3\,1\,4 \\ 8\,5\,3\,4\,1\,2\,7\,6 \end{bmatrix}$$

Let us assume that, for each student, the vector of *exchanging priorities* is $\omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

The *Exchange Places Mechanism* proceeds as follows.

(a) Let us calculate the student optimal stable matching, μ^{SO}. The application of the students-proposing deferred acceptance algorithm is the following[16]

Step	a	b	c	d
1	5, 6, 7	1, 8	2, 3	4
2	6, 7	1, 8	2, 3, 5	4
3	6, 7	1, 3, 8	2, 5	4
4	6, 7, 8	1, 3	2, 5	4
5	6, 8	1, 3, 7	2, 5	4
6	1, 6, 8	3, 7	2, 5	4
7	1, 6	3, 7	2, 5, 8	4
8	1, 2, 6	3, 7	5, 8	4
9	1, 2	3, 6, 7	5, 8	4
10	1, 2	3, 7	5, 6, 8	4
11	1, 2	3, 7	5, 6	4, 8
μ^{SO} :=	1, 2	3, 7	5, 6	4, 8

(b) Matrix Σ^{MO} is the following

$$\Sigma^{MO} = \begin{bmatrix} 3\,4\,1\,7\,5\,6\,2\,8 \\ 4\,3\,7\,5\,1\,2\,8\,6 \\ 5\,6\,3\,7\,1\,2\,4\,8 \\ 7\,8\,3\,1\,5\,6\,4\,2 \\ 1\,2\,7\,5\,3\,4\,8\,6 \\ 1\,2\,3\,7\,6\,5\,4\,8 \\ 1\,2\,4\,5\,7\,8\,3\,6 \\ 3\,4\,1\,8\,5\,6\,2\,7 \end{bmatrix} ;$$

[16] A row in the table indicates the applications that each school receives at such step. The students marked in red are the ones whose application is refused, whereas the students marked in green are tentatively accepted by the school.

(*c*) and the successive *incidence matrices* are[17]

$$I_{\mathcal{S}}^{\Sigma^{SO}} = \begin{bmatrix} 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \end{bmatrix} \text{; and thus } \mathcal{S}_1 = \{2,\, 6,\, 7,\, 8\}\,;$$

$$I_{\mathcal{S}_1}^{\Sigma^{SO}} = \begin{bmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \end{bmatrix} \text{; and thus } \mathcal{S}_2 = \{7,\, 8\}\,;$$

$$I_{\mathcal{S}_2}^{\Sigma^{SO}} = \begin{bmatrix} 1&0 \\ 1&0 \end{bmatrix} \text{; and thus } \mathcal{S}_3 = \{8\}\,.$$

Since $\mathcal{S}_3$ is a singleton, matrix $I_{\mathcal{S}_3}^{\Sigma^{SO}}$ will have a cycle containing the unique element in such a set.

Therefore, and following the order in which cycles has been reached, we have that

(1.1) student 1 will get the place that μ^{MO} assigns to student 3; the latter will take the one assigned to student 5, which will obtain the place that 1 got;

(1.2) student 4 will keep the place that μ^{MO} assigned to her;

(2.1) students 2 and 6 will exchange the seats that μ^{MO} assigned to them;

(3.1) student 7 will retain her place; and

(4.1) since no student prefers 8's place to her own, this agent will remain in her place.

To conclude this example, and with the aim of presenting a comparative for the application of some allocation procedures, relative to the data proposed in the present example, let us consider the following table. It associates each student two items: the school in which she gets a place (in blue), and the position of such school in the student's ranking (in red). Let us note that,

	Comparing Systems															
	1		2		3		4		5		6		7		8	
Boston Mechanism	b	1	c	1	c	1	d	1	d	3	a	1	a	1	b	1
Top Trading Cycles	b	1	c	1	c	1	d	1	a	1	b	2	a	1	d	4
Student Optimal	a	2	a	2	b	2	d	1	c	2	c	3	b	2	d	4
Exchange Places	b	1	c	1	c	1	d	1	a	1	a	1	b	2	d	4

[17] Each cycle in a matrix is marked by using one color (red or blue) for all the elements belonging to that cycle.

(1) The solution proposed by the *Boston* mechanism is not ε-stable. In fact, student 5 might *fairly object* this solution by proposing the student optimal stable matching. Since the last allocation is stable, no student will be able to counter-object;
(2) relative to the proposal by the *Top Trading Cycle* mechanism, and comparing it to the one suggested by the *Exchange Places* mechanism, let us observe that both allocations are efficient. The main difference can be founded in a *fairness criterion*. Let us concentrate on schools a, and b that are the ones in which both allocations differ. Note that, in the former allocation, student 6 envies students 5 and 7; and it is justifiable; for the latter allocation, 7 is the only student justifiable envying another student. She envies 5's allocation. Therefore, form a cardinal point of view, the outcome for the Exchange Places mechanism is 'fairer' than the allocation proposed by the Top Trading Cycle mechanism;
(3) finally, when comparing the *Student-Optimal Stable* and the *Exchange Places* mechanisms, it is easy to see that the latter Pareto-dominates the former. Moreover, both mechanisms induce ε-stable allocations.

7 Concluding Remarks

Let us start this section by quoting the paper by Abdulkadiroğlu et al. (2005), 'Recent Developments' Section.

> "A memorandum from Superintendent Payzant in December 2004 states that BPS plans to change the computerized process used to assign students to schools. Although the task-force report recommended that BPS adopt the TTC assignment algorithm, the School Committee is interested in simulations of both mechanisms and in understanding the extent of preference manipulation under the Boston mechanism. They are also thinking through their philosophical position on the trade-off between stability and efficiency."

This interest for defining a 'philosophical position' on the trade-off between stability and efficiency is at the origin of a modification in the mechanism used in the Boston Area, decided by July 2005, see Abdulkadiroğlu et al. (2006). As these authors mention, the solution was to adopt a deferred acceptance mechanism because it is strategy-proof. Nevertheless, as we have pointed out in the present paper, this solution is far from solving the trade-off which is at the origin of this reform. In fact the employ of such a solution can be justified because it considers that stability is the central issue. If, moreover, the best that agents can do is to reveal their true characteristics, it is straightforward to conclude, as the Boston School Committee did, that the Student-Optimal Stable mechanism would be adopted.

The main conclusion of this paper is to provide two ways for avoiding the efficiency-equity dilemma. The first one, that allows to slightly modify the last reform introduced by the Boston School Committee, comes from redefining how the schools influence on designing which students are prioritized. In this sense, the solution comes from the employ of scoring rules that, applied to the students' characteristics, fulfill our *limited freedom condition.*

The second approach to escape to this trade-off between stability and efficiency comes from a reinterpretation of the instability notion. In this sense, the additional reform that would be introduced in systems focusing on stability is allowing students to exchange the places that were allocated to them in the actual system.

Thus, when the Public authority does not want to restrict the schools freedom when prioritizing students, the *Exchange Places Mechanism* can be introduced to minimize the (inevitable) trade-off between efficiency and equity.

A further aspect that can be used to promote the use of the *Exchange Places Mechanism* comes from the arguments given in Abdulkadiroğlu et al. (2006). What these authors report, concerning the recent changes introduced by the Boston School Committee, is:

> "As far as we know, it is the first time that "strategyproofness," a central concept in the game theory literature on mechanism design, has been adopted as a public policy concern related to transparency, fairness, and equal access to public facilities." (Abdulkadiroğlu et al. (2006), pg. 2.)

Nevertheless, strategy-proofness was not considered as a 'sufficient condition' justifying a modification in the mechanism employed to allocate schools places among students. In particular, the Top Trading Cycle mechanism, introduced by Abdulkadiroğlu and Sönmez (2003) was not considered by the Boston School Committee as a satisfactory proposal to modify the former system. This mechanism is strategy-proof and selects efficient allocations. When comparing the Top Trading Cycle and the student-proposing deferred acceptance mechanisms, which is also strategy-proof, Abdulkadiroğlu et al. (2006) conclude the following:

> "While TTC is a Pareto efficient mechanism when only students are considered, and the student-proposing deferred acceptance mechanism is not, the former does not Pareto dominate the latter. One implication is, based on a stronger efficiency notion (such as a cardinal efficiency notion relying on the rank order of schools) the student-proposing deferred acceptance mechanism may perform better than the TTC for some problems. For example, the student-proposing deferred acceptance mechanism may assign more students to their first choices than TTC. Moreover while each Nash equilibrium outcome of the complete information preference revelation game induced by the Boston mechanism is weakly Pareto dominated by each dominant-strategy equilibrium outcome of the student-proposing deferred acceptance mechanism (Ergin and Sönmez (2006)), equilibrium outcomes induced by the Boston mechanism and TTC are not Pareto ranked." (Abdulkadiroğlu et al. (2006), pg. 10.)

Given the above comparison, we can conclude that our *Exchange Places Mechanism* performs better than all the three mechanisms considered in the literature, namely the TTC, the student-proposing deferred acceptance and the Boston mechanisms. The reason is that, provided the arguments by Abdulkadiroğlu et al. (2006), if take into account that, since the *Exchange Places Mechanism* Pareto dominates the student-proposing deferred acceptance, we have that:

(a) The *Exchange Places Mechanism* never assigns less students to their first choices than the *student-proposing deferred acceptance* mechanism do, and (*b*) when students do not act strategically, the *Exchange Places Mechanism* weakly Pareto dominates the student-proposing deferred acceptance mechanism.

Acknowledgments. The authors acknowledge Josep E. Peris and Pablo Revilla for the many discussions we had around this paper. We would also like to acknowledge the financial support by the Institut Valencià d'Investigacions Econòmiques, FEDER and the Spanish Ministerio de Educación y Ciencia under project SEJ2007-64649.

Appendix

I. A Proof for Proposition 3.4

To prove Proposition 3.4, let us consider the following School Allocation Problem. $\mathcal{S} = \{1,2,3\}$; $\mathcal{C} = \{a,b,c\}$; $Q = (1,1,1)$; and the ranking and priorities matrices are

$$\Phi = \begin{bmatrix} 1\,3\,2 \\ 2\,3\,1 \\ 2\,3\,1 \end{bmatrix}, \text{ and } \qquad \Pi = \begin{bmatrix} 3\,2\,1 \\ 2\,1\,3 \\ 1\,3\,2 \end{bmatrix}$$

Note that in such a problem there is only one stable matching, μ, such that $\mu(1) = c$; $\mu(2) = b$; and $\mu(3) = a$. Nevertheless, μ fails to be Pareto efficient since μ' defined as $\mu'(1) = a$; $\mu'(2) = b$; and $\mu'(3) = c$, Pareto dominates μ. ■

II. A Proof for Propositions 4.1 and 4.3

To prove propositions 4.1 and 4.3, let us start by establishing some claims that will be helpful to understand the structure of stable and efficient matchings.

Claim 1 Let $\mathcal{SAP}$ be a School Allocation Problem. Then it has, at most, one stable and efficient matching.

Proof. First of all, and following Martínez et al. (2001), let us note the set of stable matchings has a lattice structure. The operators proposed by Martínez et al. (2001) to prove such a structure are just the students' rankings and the schools' priorities. Therefore, if a stable matching μ is efficient for $\mathcal{SAP}$, it must be the *student optimal stable matching.* ■

Claim 2 Let $\mathcal{SAP}$ be a School Allocation Problem, and μ be a stable matching for it. Then μ is efficient if, and only if, there is no (non-empty) ordered set of students

$$\left\{s^1, \ldots, s^i, \ldots, s^k\right\} \subseteq \mathcal{S}, \text{ such that}$$

(*a*) $\mu\left(s^i\right) \in \mathcal{C}$ for all $s^i \in \mathcal{S}$, and

(*b*) each student s^i ranks her mate as worse than her *next-in-the-order* student's mate,

$$\rho_{s^i\mu(s^{i+1})} < \rho_{s^i\mu(s^i)} \text{ for each } i < k, \text{ and } \rho_{s^k\mu(s^1)} < \rho_{s^k\mu(s^k)}.$$

Proof. First of all, let us note that if there exists a set of students fulfilling the statement of Claim 2, it is easy to see that the matching is not efficient. To see that, given a matching μ, let us construct μ' such that $\mu'(s_i) = \mu(s_i)$ for each $s_i \notin \{s^1, \ldots, s^i, \ldots, s^k\}$ and for any student s^i in such a set, $\mu'(s^i) = \mu(s^{i+1})$, modulo k. Note that μ' Pareto dominates μ.

On the other hand, let us assume that μ is a stable matching that fails to be efficient. Then, there should be another matching μ' that dominates μ. This is equivalent to say that $\mu' \neq \mu$ and, for each student s_i such that $\mu'(s_i) \neq \mu(s_i)$, $\rho_{s_i \mu'(s_i)} < \rho_{s_i \mu(s_i)}$.

Let s^1 be a student such that $\mu'(s^1) \neq \mu(s^1)$. Note that, by the Decomposition Lemma,[18] and provided that μ is stable, it must be the case that $\mu(s^1) \neq s^1$. Let denote $\mu'(s^1) = c^1 \in \mathcal{C}$. Since μ is stable, and $\rho_{s^1 c^1} < \rho_{s^1 \mu(s^1)}$, it must be the case that $|\mu(c^1)| = q_{c^1}$. Therefore, there should be a student, say $s^2 \in \mu(c^1) \setminus \mu'(c^1)$. Since μ' Pareto dominates μ, and $\mu(s^2) \neq \mu(s^2)$, there should be an school, say c^2 such that $\mu'(s^2) = c^2$. Note that if $\mu(s^1) = c^2$ the result is proved. Otherwise, there should be a student, say $s^3 \in \mu(c^2) \setminus \mu'(c^2)$. Let us observe that if $\mu'(c^3) = \mu(c^1)$, the result follows. Otherwise, since the set of students is finite, an iterative argument yields the desired result. ∎

We can now proceed to prove propositions 4.1 and 4.3.

Proof of Proposition 4.1.

Let $\mathcal{SAP}$ be a School Allocation Problem. Let us assume that Π satisfies *CPC*.

By Claim 1 we now that, if there is a stable and Pareto efficient matching, it must be μ^{SO}, the student optimal stable matching. Since μ^{SO} is stable, let us assume that it is not efficient.

Then, by Claim 2, there must be a matching μ', and an ordered set of students, say $\{s^1, \ldots, s^i, \ldots, s^k\} = \mathcal{S}'$, such that for each $s^i \in \mathcal{S}'$, $\rho_{s^i \mu'(s^i)} < \rho_{s^i \mu^{SO}(s^i)}$, with $\mu'(s^i) = \mu^{SO}(s^{i+1})$ for each $i \leq k-1$, and $\mu'(s^k) = \mu^{SO}(s^1)$.

Since μ^S is stable, we have that, for each student $s^i \in \mathcal{S}'$,

(*a*) $\pi_{\mu^{SO}(s^{i+1}) s^i} > q_{\mu^{SO}(s^{i+1})}$; and

(*b*) $\pi_{\mu^{SO}(s^{i+1}) s^i} > \pi_{\mu^{SO}(s^{i+1}) s^{i+1}}$.

where $s^{k+1} = s^1$.

By *CPC*, there must be a priorities order π such that, for each school $c_j \in \mathcal{C}$, and any two students s_i, s_h such that $\min\{\pi_{c_j s_i}, \pi_{c_j s_h}\} > q_{c_j}$, it holds that

$$\left[\pi_{c_j s_i} < \pi_{c_j s_h}\right] \Leftrightarrow \left[\pi_{s_i} < \pi_{s_h}\right].$$

Therefore, by *CPC* and stability of μ, we have that

$$\pi_{s^1} < \ldots < \pi_{s^i} < \pi_{s^{i+1}} < \ldots < \pi_{s^k} < \pi_{s^1},$$

which contradicts that π might represent a priorities order. ∎

[18] See, for instance Gale and Sotomayor (1985). Note that Martínez et al. (2000) pointed out that this results is still valid in our framework.

Proof of Proposition 4.3.

Let $\mathcal{SAP}$ be a School Allocation Problem. Let us assume that Φ satisfies *CRC*, relative to ρ, which is the ranking that all the students exhibit. Note that, in such a case it is straightforward to see that there is a unique stable matching, which is also Pareto efficient.

This matching can be obtained in a simple sequential way. If we denote by $c^t \in \mathcal{C}$ the t-th college according ρ, we proceed as follows:

(1) $\mu^S\left(c^1\right) = \{s_i \in \mathcal{S}: \pi_{c^1 s_i} \leq q_{c^1}\}$;

(2) $\mu^S\left(c^2\right)$ is the set containing the q_{c^2} prioritized students, according to π_{c^2}, that are not in $\mu^S\left(c^1\right)$; ...

(t) $\mu^S\left(c^t\right)$ is the set containing the q_{c^t} prioritized students, according to π_{c^t}, that are not in $\mu^S\left(c^h\right)$ for any $h < t$.

It is easy to see that this assortative matching is both stable and efficient. ■

III. A Proof for Theorem 4.9

To prove Theorem 4.9, let us consider a School Allocation Problem. Let us assume that there is a 'scoring function' $GS : \mathcal{S} \times \mathcal{C} \to \mathbb{R}_+$ inducing the priorities matrix Π. Note that this is equivalent to say that for each $c_j \in \mathcal{C}$, and any two students s_i and s_h in $\mathcal{S}$, $\pi_{c_j s_i} < \pi_{c_j s_h}$ whenever $GS\left(s_i, c_j\right) > GS\left(s_h, c_j\right)$.

Let us assume that GS satisfies LFC. Then there are three functions $LS : \mathcal{S} \to \mathbb{R}_+$, $P : \mathcal{C} \to 2^{\mathcal{S}}$, and $CS : \mathcal{S} \times \mathcal{C} \to \mathbb{R}_+$ decomposing GS as established in Definition 4.8.

For function LS given, let us define the 'common priority' π as

$$\pi_{s_i} = \left|\{s_h \in \mathcal{S} :\ LS\left(s_h\right) \geq LS\left(s_i\right)\}\right|.$$

Then, since GS satisfies LFC, and, for each $c_j \in \mathcal{C}$, π_{c_j} is induced by LFC, it follows that Π satisfies CPC related to π. Thus, as Proposition 4.1 establishes, $\mathcal{SAP}$ has a unique stable and efficient matching. ■

IV. A Proof for Theorem 6.5

To prove Theorem 6.5, let us consider a School Allocation Problem, $\mathcal{SAP}$, and let μ^{SO} its student optimal stable matching. By Martínez et al. (2001), we know that for any matching μ', if $\rho_{s_i \mu'(s_i)} < \rho_{s_i \mu^{SO}(s_i)}$ for some student c_i, then μ' is unstable.

Now, let Σ^{SO} be a matrix μ^{SO}-agreeing Φ, and μ^{TTC} the matching obtained by applying the μ^{SO}-Σ^{SO}-*TTC* algorithm. By Alcalde-Unzu and Molis (2009), Theorem 4, we have that μ^{TTC} is efficient. Moreover, Alcalde-Unzu and Molis (2009), Corollary 3, also establishes that, for each student s_i,

$$\rho_{s_i \mu^{TTC}(s_i)} \leq \rho_{s_i \mu^{MO}(s_i)}.$$

Now, let us assume that μ^{TTC} is not ε-stable. Since it is efficient, it should fail to be φ-stable. Therefore, there should be a student, s_i, that can *fairly object* μ^{MO} via some matching, say μ'. Then, μ' must satisfy that

$$\rho_{s_i \mu'(s_i)} < \rho_{s_i \mu^{TTC}(s_i)} \leq \rho_{s_i \mu^{MO}(s_i)},$$

which implies that μ' is unstable. Note that this instability implies that there should be a student, say s_h, and matching μ'' such that s_h can counter-object μ' via μ''. A contradiction. ■

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