Bidding for Being a Seller*

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Abstract

We propose a new procedure for supplying Public Firms. This mechanism is inspired in an auction allocating each bidder a share of the total amount to be supplied. The fact that this share is endogenously determined could induce some nice properties. First, bidders have a *true* incentive to participate in the procedure. Second, the Public Firm's expense could be lower than what it can obtain in a classical Vickrey auction. Third, the Public Firm could have an active role on the design of the particular way in which the share is decided.

When concentrated in the two-bidder case, we show that a unique Nash equilibrium is reached.

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JEL Classification Numbers:

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1. Introduction

Public Firms use to buy some goods by means of auction-like processes. The structure of this system is the following. First, the Public Firm announces which good it wants to buy, the amount of such a good and its budget constraint (i.e. how much it is able to spend). Secondly, its potential providers announce their proposal on the price at which they could provide that good. Finally, the provision of the good is ordered to the provider asking for the lower price. This supplying procedure is not only employed by public firms, but also by (large-sized) firms.

This procedure, which resembles a first price auction, is employed in several situations. For instance, Hospitals (or Public Health Offices) use to employ such a procedure to buy different elementary articles such as medicines and/or surgical items; the Army also employs this kind of procedures to buy some armaments such as bullets, guns; and military clothes.

Just to point out how important this system is, let us note that, following the Health Industry Distributors Association (HIDA), "in total, an average of 94.8 million seasonal flu vaccine doses have been produced for the United States each year since 2000," [4, pg. 5.]

The origin of such a mechanism can be found in two assumptions. The first one is that any provider can supply as much quantity of good as desired; and the second one is that this mechanism induces a (Bertrand's) competition among the potential providers that leads the lower price that the Public Firm could guarantee.

Nevertheless, there is evidence showing that, in particular, the first hypothesis is sometimes unrealistic. The fact that this assumption was not fulfilled could have unrecoverable effects in some particular situations. Think of the unavailability of flu vaccines during the 2004-2005 influenza season, or the general difficulties to obtain Prevnar doses during the last years.

In particular, the shortage of flu vaccines seems to be related to the fact that one of the two companies that supplies fu vaccine to the American market suspended its production activity. In fact, Chiron Corp. announced that the British government suspended the manufacturing license in Liverpool for three months, citing contamination problems.[1]

A feature which is important comes from the usual real-live timing of the relevant facts. The decision of which firm will be the (unique) provider is taken some months before the good should be delivered. This is because the production process involves the employ of a time which is increasing on the quantity to be produced. For instance, following the above-mentioned inform by HIDA [4, pg. 9.], "the manufacturing process is tipically five to eight months long. It is subject to delays depending on the results of virus selection and incubation processes." As a consequence of the above, we have that if the unique provider (partially) fails to deliver the Public Firm's needs, it is hardly

[1] This information was circulating by most of the US media in October 2004. See, for instance the CNN's new at http://www.cnn.com/2004/ALLPOLITICS/10/19/bush.flu/index.html

difficult to find an *alternative* provider to complete the under-supplying at time. Note that, when this problem concerns vaccines, this situation could have undesirable effects on the population's health.

Clearly, a possible solution to the under-supplying problem can be founded by increasing the number of firms which will provide the desired good, by determining how the total amount of good is shared among the providers. Nevertheless, if providers' quotae are exogenously determined, this solution could raise excessively the Public Firm's expense.

Our proposal is to design allocation rules based on what we call the *serial allocation procedure*. The input for these rules will be similar to that in real-live auction process, namely the agents' bids (or providers' proposed prices). The main difference with the usual auctions is the output for the process. Our approach is to assign each agent a share on the quantity of good to be bought by the Public Firm. Therefore, what we propose in this paper is not to study some well-known auctions procedures but to employ new allocation rules in which:

(a) The Public Firm could play an active role on designing it,

(b) (some of) the providers might have true incentives to participate in the allocation process; and

(c) the Public Firm's expected expense could be lower than what it can obtain in a classical Vickrey auction.

Let us justify the interest of the above facts. First, in the literature of auction mechanisms,[2] auctioneers do not play an active role. In fact, most of the authors consider that the allocation procedure is exogenously given: The seller offers the object to be auctioned and receives the price paid by the bidder winning the auction. Then, no strategic behavior is assumed to the auctioneer. In fact, as Alcalde et al. [2] pointed out, if agents are well-informed and the auctioneer is not constrained to reveal her true reservation value, she will get all the surplus. In this paper, we are not going to consider that the Public Firms play strategically when informing on their reservation values;[3] but they could have an active role when selecting the particular procedure for allocating the share of their demand that will be covered by each provider.

The second aspect that we want mention is related to assertion (b) above. Let us notice that, provided that the unique firm that covers the Public Firm's demand is the one having the lowest cost (and thus proposing the lowest price), there is not reason justifying the presence, at the allocation process, of the other firms. Nevertheless, their presence is crucial to reach the results. Therefore, the question that each firm well might propose itself is: If I guess that my proposal will not be considered, why should I participate in this procedure? Moreover, if every firm is free to participate in this

[2] The reader can find an great survey in the (today young, but) classic books by Klemperer [5] or Krishna [6].

[3] In fact, this reservation value could be also interpreted as the *market value* of the object to be sold. Note that, under such an interpretation, no strategic behavior might be assumed to the Public Firms.

process or not, it could be expected that just one firm will be present in the allocation process, which might be in contradiction with the existence of an equilibrium. Our approach avoids such considerations by allowing that (at least) two agents will have a positive share on the provision of the good demanded by the Public Firm, when their average cost does not exceed the Public Firm's reservation price.

Relative to assertion (c) above, let us note that the classical auction procedures (e.g. the simultaneous first-price sealed auction, and the sequential Dutch and English auctions) induce a Bertrand's competition among the bidders. This kind of competition is the main (intuitive) reason why the Public Firm pays the minimal price that the provider with the second lowest cost can propose. This price coincides with the one settled, at equilibrium, in the classical second-price Vickrey auction. Nevertheless, there is no incentive for the provider having the lowest cost to propose a price very different to the one that her nearest opponent could propose. The family of procedures that we introduce in this paper induces the agents to solve a trade-off between reaching a higher share of the object to be sold (by decreasing the suggested price) and obtaining a higher mark-up in their sales (by increasing the suggested price). By introducing this new trade-off into agents' consideration we might ensure a net benefit for the auctioneer, related to that reached in the classical auction mechanisms (i.e. the second lowest cost).

The way in which we provide answers to our questions is the following. First, Section 2 introduces the model and the main definitions. Section 3 proposes an allocation procedure for the two-agent case, in which both sellers will have incentives to participate actively. Section 4 proposes a way to build serial allocation procedures, and show the existence of a unique Nash equilibrium for each such mechanism. Section 5 explores the existence of mechanisms in which the average price is lower than the second lowest cost. Section 6 introduces how to deal when more than two providers compete at the process. Section 7 proposes a mechanism in which agents, having private information about their own characteristics, select a price similar to the one expected in the serial procedure. The main interest of such a mechanism is that providers do not need any extra information about their opponents' costs. Finally, Section 8 presents the main conclusions and some open questions.

2. The model

Let us consider a Public Firm, to be denoted agent 0, that wants to buy a certain quantity of a (perfectly divisible) good. This firm faces a budged constraint B_0. This quantity of good can be (totally of partially) provided by a set of agents, to be called the sellers or providers. Let $\mathcal{I} = \{1, \ldots, i, \ldots, n\}$ be the set of potential providers. We assume throughout the paper that n, the number of sellers, is finite and greater that one. Provider i faces a total cost of c_i for fully providing the good.

When a seller, say i, provides a share π_i of the total quantity asked by the Public

Firm its profits are

$$\Pi_i = \pi_i (p_i - c_i)$$

being p_i the price it asked for fully providing agent 0's demand.

Relative to the Public Firm, there are two functions that are particularly relevant. The first one will measure the savings it reaches when a particular way to share its demand is selected, and the second one is the utility it obtains, derived from a particular share. To introduce these functions, for B_0 given, let us consider the pair (π, P) where $\pi = (\pi_1, \ldots, \pi_i, \ldots, \pi_n)$ is the share vector (i.e. $\pi_i \geq 0$ for all i, and $\sum_{i=1}^{n} \pi_i = 1$), and $P = (p_1, \ldots, p_i, \ldots, p_n)$ is the vector summarizing the prices selected by the sellers. We denote by S_0 the Public Firm's savings function, described by

$$S_0 (B_0, \pi, P) = \sum_{i=1}^{n} \pi_i (B_0 - p_i) .$$

In a similar way, we define Public Firm's utility function, described by

$$U_0 (B_0, \pi, P) = \left[\sum_{i=1}^{n} (\pi_i (B_0 - p_i))^{\rho} \right]^{\frac{1}{\rho}}, \text{ where } 0 < \rho < 1.$$

Note that, in order to capture the aim of providers' security argued in the Introduction, it could be useful to assume that the Public Firm has a preference for variety given by a CES utility function. Therefore, if we assume that $\sigma > 1$ is the (constant) elasticity of substitution, we reach the above expression for $\rho = 1 - \frac{1}{\sigma}$.

Notice that when $\sigma \to \infty$, $U_0(B_0, \pi, P)$ tends to $S_0(B_0, \pi, P)$. On the other hand, when σ is smaller, indifference curves are strictly convex. Therefore, for the same per capita saving, two providers with the same market share are preferred to only one.

The problem we face with is how to determine each seller's share. For, we propose a general formulation by defining what we will call an *allocation procedure*.

Definition 2.1. *An allocation procedure is a function*

$$\Psi : \mathbb{R}^{n+1} \to \mathbb{R}^n$$

such that for each B_0*, and any* $P \in \mathbb{R}^n$*,* $\Psi(B_0, P)$ *is such that, for each* i *in* $\mathcal{I}$*,* $0 \leq \Psi_i(B_0, P)$*, and* $\sum_{i=1}^n \Psi_i(B_0, P) = 1$.

For fairness purposes, we concentrate on allocation procedures satisfying two requirements. The first one is that no agent proposing a price higher than the Public Firm's budget constraint will have a positive share on the provision; whereas the second one is that agents' shares on the provision should be responsive to their announced prices. Formally,

(*i*) For all $P \in \mathbb{R}^n$, each $i \in \mathcal{I}$, if $p_i > B_0$ then $\Psi_i(B_0, P) = 0$,[4] and

[4] To avoid the possibility of *degenerated* provision proposals, we do consider price vectors P having at least an acceptable proposal. I.e. we identify the dominium of Ψ with the set of n-dimensional vectors P such that $p_i < B_0$ for at least an agent i in $\mathcal{I}$.

(ii) For all $P \in \mathbb{R}^n$, each i and j in $\mathcal{I}$, if $p_i \leq p_j$ then $\Psi_i(B_0, P) \geq \Psi_j(B_0, P)$.

Under our considerations, if P is the vector summarizing the sellers' proposed prices, provider i's payoff is $U_i(B_0, P) = \Psi_i(B_0, P)[p_i - c_i]$, which coincides with its profit.

Given an allocation procedure Ψ we can describe a normal-form game $\Gamma = \{\mathcal{I},\ S,\ U,\ \Psi\}$ in which the set of agents is $\mathcal{I}$, each agent's strategy space is the interval $[0, B_0]$ and each provider's utility follows the expression,

$$U_i(B_0, P) = \Psi_i(B_0, P)[p_i - c_i].$$

3. An Allocation Procedure for Two Buyers

In this section we propose an allocation procedure for two agents, i.e. $\mathcal{I} = \{1,\ 2\}$. The particular market sharing function we are going to employed is inspired in an old proposal to bankruptcy problems known as the *Contested Garment Principle.* (See Dagan [3] for an analysis of this bankruptcy solution). What is important to stress is that, from a normative point of view, it fulfills very nice characteristics. In particular, it not only satisfies the two properties proposed in the previous section, but it is also continuous. Moreover, from a positive point of view, as we will see, there exists a unique Nash equilibrium.

Let us introduce the allocation procedure that we call Ψ^{cg}.

Definition 3.1. *We define the allocation procedure* Ψ^{cg} *as the function that, for each* B_0*, and* $P = (p_1, p_2)$ *associates to provider* i*,* $i = 1, 2$*, the share*

$$\Psi_i^{cg}(B_0, P) = \frac{B_0 - 2p_i + \max\{p_1, p_2\}}{2[B_0 - \min\{p_1, p_2\}]}$$

Let us remember that, for the shake of concreteness, we assume that the prices proposed by the agents are restricted to not exceed the Public Firm's budget B_0. We must also mention that, when $p_1 = p_2 = B_0$, each provider's share is one half.

Our first result lies on the uniqueness of Nash Equilibrium for the game associated to Ψ^{cg}, namely $\Gamma^{cg} = \{\mathcal{I}, S, U, \Psi^{cg}\}$. Since this result is a corollary of Theorem 4.2, its proof is omitted.

Theorem 3.2. *Let* $\mathcal{I} = \{1, 2\}$ *be the set of providers. Let us assume that* $0 \leq c_1 \leq c_2 < B_0$*. Then, there is a unique Nash equilibrium* P^* *for* Γ^{cg} *described by*

$$\begin{aligned} p_1^* &= B_0 - \frac{(B_0 - c_1)^{\frac{1}{2}} (B_0 - c_2)^{\frac{1}{2}}}{2}, \text{ and} \\ p_2^* &= \frac{B_0 + c_2}{2} \end{aligned}$$

Let us observe that, at equilibrium, provider 2's utility is positive. Moreover, the

Public Firm's expected saving is[5]

$$S_0\left(B_0, P^*\right) = \frac{B_0 - c_2}{4}\left[2\left(\frac{B_0 - c_1}{B_0 - c_2}\right)^{\frac{1}{2}} + \left(\frac{B_0 - c_2}{B_0 - c_1}\right)^{\frac{1}{2}} - 1\right].$$

Note that, in particular, if

$$\frac{B_0 - c_1}{B_0 - c_2} \geq \left(\frac{21}{8} + \frac{5}{8}\sqrt{17}\right) \approx 5.2019$$

it holds that $S_0\left(B_0, P^*\right) > B_0 - c_2$.

As a consequence of the above facts we can stablish that, when B_0 is close enough to c_2, and $c_2 > c_1$, the Public Firm would prefer to employ Ψ^{cg} rather than to use a second-price buyed Vickrey mechanism.

Corollary 3.3. *Let us assume that* $0 \leq c_1 < c_2 < B_0$*, and let* P^* *be the unique equilibrium for* Γ^{cg}*. If* $8\left(c_2 - c_1\right) \geq \left(13 + 5\sqrt{17}\right)\left(B_0 - c_2\right)$ *then*

$$U_0\left(B_0, \Psi^{cg}\left(B_0, P^*\right), P^*\right) > B_0 - c_2$$

[5] Throughout the paper we will abuse notation, and for a given allocation procedure, say Ψ, we will write $S_0\left(B_0, P\right)$ instead $S_0\left(B_0, \Psi\left(B_0, P\right), P\right)$, and $U_0\left(B_0, P\right)$ will be used to refer $U_0\left(B_0, \Psi\left(B_0, P\right), P\right)$.

4. Two Agents' Allocation Procedures and the Buyer's Role

The previous section pointed out that allocation procedures could improve both, the Public Firm's savings and the Public Firm's utility, related to what it should expect in a *bought lower-price auction*. The reason in very simple. In the two-seller case, the firm with highest cost has a true incentive to propose a price lower than the Public Firm's budget, B_0. Therefore, it will get a positive share on the provision of the needs by the Public Firm. Then, what the seller having the lowest cost should investigate is how to reduce its opponent's share. The answer is very simple: Its only possibility comes from stating a low price. Therefore, the firm having the lowest cost should solve its trade-off between a low price (inducing a high share) and a high price (inducing a high mark-up on its share). Then, the Public Firm might explore the possibility of designing an allocation procedure guaranteeing an expected saving higher than $B_0 - c_2$, i.e. the saving that it can guarantee by using a Vickrey auction.

We now propose a general description of allocation procedures, for the two-agent case, satisfying some basic properties:

(a) Anonymity: Each seller's share on the provision is independent of its label. It only depends on the price it sets and the proposed by its rival. Anonymity is described as $\Psi_1(B_0; p_1, p_2) = \Psi_2(B_0; p_2, p_1)$ for any p_1 and p_2.

(b) Continuity: The sharing function Ψ is continuous whenever $(0,0) < (p_1, p_2) <$

(B_0, B_0).

(c) Monotonicity: Each seller's share is strictly decreasing in its proposed price. Formally, for each $0 \leq p_1 < p_1' \leq B_0$, and any $0 \leq p_2 < p_2' \leq B_0$,

$$\Psi_1(B_0; p_1, p_2) > \Psi_1(B_0; p_1', p_2); \text{ and } \Psi_2(B_0; p_1, p_2) > \Psi_2(B_0; p_1, p_2').$$

(d) Homogeneity: The shares assigned by Ψ are independent of the numéraire employed, i.e., for all P, and each positive scalar δ

$$\Psi(B_0, P) = \Psi(\delta B_0, \delta P).$$

The next result proposes a simple way to build allocation procedures for two-bidders.

Lemma 4.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function such that $f(0) = 0$. Then the allocation procedure Ψ^f described as*

$$\Psi^f(B_0; P) = \begin{cases} \left(\frac{f(B_0 - p_1)}{2f(B_0 - p_2)}, \frac{2f(B_0 - p_2) - f(B_0 - p_1)}{2f(B_0 - p_2)}\right) & \text{if } p_2 \leq p_1 < B_0 \\ \\ \left(\frac{2f(B_0 - p_1) - f(B_0 - p_2)}{2f(B_0 - p_1)}, \frac{f(B_0 - p_2)}{2f(B_0 - p_1)}\right) & \text{if } p_1 \leq p_2 < B_0 \end{cases}$$

is an anonymous, continuous and monotonic allocation procedure. Moreover, if there is *some $\alpha > 0$ such that f is homogeneous of degree α,*[6] *Ψ^f satisfies homogeneity.*

The above result is helpful to demonstrate the existence of equilibrium for each two-agent allocation procedure that could be expressed as in Lemma 4.1. Moreover, if the function f above is homogeneous,[7] the unique equilibrium can be characterized in a easy way.[8] Let us assume that $c_1 \leq c_2 < B_0$; to compute the unique Nash equilibrium we can proceed as follows. First, seller 2 computes its optimal price conditioned to being not lower than that proposed by its opponent

$$\begin{aligned} p_2^* = \quad & \arg\max U_2\left(B_0; p_1, p_2\right) \\ & s.t.\ p_1 \leq p_2 \end{aligned} \tag{4.1}$$

What it is important to stress is that p_2^* is independent of p_1^*, except on the fact that the latter must be not higher than p_2^*. Then, once seller 2's optimal price is anticipated, its opponent computes its optimal price, under the assumption that it does not exceed p_2^*

$$\begin{aligned} p_1^* = \quad & \arg\max U_1\left(B_0; p_1, p_2^*\right) \\ & s.t.\ p_1 \leq p_2^* \end{aligned} \tag{4.2}$$

[6] *The notion of homogeneity of degree α is very standard. We say that function $g : \mathbb{R}^\ell \to \mathbb{R}$ is homogeneous of degree α if for all x in $\mathbb{R}^\ell$ and positive scalar λ,*

$$g\left(\lambda x\right) = \lambda^\alpha g\left(x\right)$$

[7] Throughout the paper we will say that a function f is homogeneous if there is some positive scalar α such that f satisfies homogeneity of degree α.

[8] For the technical details, see Appendix 1.

The above arguments are summarized in Theorem 4.2, and detailed explained in Appendix 1, which is devoted to prove that result.

Theorem 4.2. *Let f be a degree α homogeneous, $\alpha > 0$, real-valued function such that $f(0) = 0$. Let Ψ^{α} the allocation procedure induced by f. Then the two-agent game $\Gamma^{\alpha} = \{\mathcal{I}, S, U, \Psi^{\alpha}\}$ has a unique Nash Equilibrium.*

We next deal with a robustness analysis of the equilibrium. What we will see is that, when agents are assumed not to employ dominated strategies, and we introduce an iterative procedure of deleting dominated strategies, this procedure converges to a unique strategy for each player. This strategy is the one employed at the unique equilibrium mentioned in Theorem 4.2. Using Moulin's [7] terminology, we will see that the game that we proposed is dominance solvable.

To reach our objective, we need an additional assumption. Just to introduce it let us consider α to be given, and modify slightly the game Γ^{α} by introducing $\Gamma^{\alpha d} = \left\{\mathcal{I}, S, U, \Psi^{\alpha d}\right\}$, where the new allocation procedure $\Psi^{\alpha d}$ is described by

$$\Psi^{\alpha d}(B_0; p_1, p_2) = \begin{cases} \left(1 - \frac{(B_0 - p_2)^{\alpha}}{2(B_0 - p_1)^{\alpha}}, \frac{(B_0 - p_2)^{\alpha}}{2(B_0 - p_1)^{\alpha}}\right) & \text{if } p_1 \leq p_2 \leq d \\ \left(\frac{(B_0 - p_1)^{\alpha}}{2(B_0 - p_2)^{\alpha}}, 1 - \frac{(B_0 - p_1)^{\alpha}}{2(B_0 - p_2)^{\alpha}}\right) & \text{if } p_2 \leq p_1 \leq d \\ (1, 0) & \text{if } p_1 \leq d < p_2 \\ (0, 1) & \text{if } p_2 \leq d < p_1 \\ (0, 0) & \text{if } d < \min\{p_1, p_2\} \end{cases}$$

The intuition behind the employ of $\Psi^{\alpha d}$ is the following.[9] Let us imagine that there is a price d, close enough to B_0, that all the agents agree that will be exceeded by no provider. For instance, let us imagine that $B_0 = 1000000.01\ USD$, and it is known (also by the Public Firm) that no agent will ask more that 1 million USD. Given that, what $\Psi^{\alpha d}$ does is to allocate a null share to each agent exceeding d, in such a way that if both agents exceed d, the Public Firm will decide not to buy anything. Let us notice that, even do the games Γ^{α} and $\Gamma^{\alpha d}$ are not the same, given that d might be arbitrarily close to B_0, the general conclusions obtained for $\Gamma^{\alpha d}$ could also be employed to interpret what is happening when agents play the game Γ^{α}. The next result, whose proof is relegated to Appendix 1, establishes that the equilibrium obtained in Γ^{α} is robust, in the sense above-mentioned, when d, is close enough to B_0.

Theorem 4.3. *The game* $\Gamma^{\alpha d} = \left\{\mathcal{I},\ S,\ U,\ \Psi^{\alpha d}\right\}$ *is dominance solvable.*

5. Auction versus Allocation Procedures. A Buyer's Decision

In the previous section we pointed out that the Public Firm can have an active role when selecting how the sellers will share the quantity to be sold. The aim of this section is to extend the discussion relative to the buyer's decision. For interpretative purposes, we will concentrate on a family of homogeneous allocation procedures, to be called α-serial

[9] Even do Theorem 4.3 is true for any $d < B_0$, for interpretative pourposes we will consider that d is close enough to B_0.

mechanisms

$$\Psi^{\alpha}\left(B_0;P\right)=\begin{cases}\left(\frac{(B_0-p_1)^{\alpha}}{2(B_0-p_2)^{\alpha}},\frac{2(B_0-p_2)^{\alpha}-(B_0-p_1)^{\alpha}}{2(B_0-p_2)^{\alpha}}\right) & \text{if } p_2\leq p_1<B_0\\ \\ \left(\frac{2(B_0-p_1)^{\alpha}-(B_0-p_2)^{\alpha}}{2(B_0-p_1)^{\alpha}},\frac{(B_0-p_2)^{\alpha}}{2(B_0-p_1)^{\alpha}}\right) & \text{if } p_1\leq p_2<B_0\end{cases}$$

To formally complete the description of Ψ^{α}, let us assume that $\Psi^{\alpha}(B_0;P)=(1,0)$ (resp. $(0,1)$) if $p_1<p_2=B_0$ (resp. $p_2<p_1=B_0$), and $\Psi^{\alpha}(B_0;P)=(1/2,1/2)$ whenever $p_1=p_2=B_0$.

Note that it is easy to see that Ψ^{α} can be generated by any function f, as the employed in Theorem 4.2, homogeneous of degree α. The employ of such functions can help us to understand the intuition behind the selection of a particular α. Note that such a parameter can be interpreted as the *price elasticity* of a seller's share.

Provided that the parameter α is chosen by the Public Firm, it could be interesting to explore how its savings are related to α. Just to provide an intuition on this relationship, it could be useful to notice that the solution of problem (4.1) is[10]

$$p_2^*=\frac{B_0+\alpha c_2}{\alpha+1} \tag{5.1}$$

[10] Equations (5.1) and (5.2) could be obtained from the standard first order conditions relative to programs (4.1) and (4.2) respectively.

and that the solution of problem (4.2) must satisfy

$$(B_0 - p_1^*) = \frac{1}{2}\left(\frac{B_0 - p_2^*}{B_0 - p_1^*}\right)^{\alpha}[B_0 + (\alpha - 1)\, p_1^* - \alpha c_1] \tag{5.2}$$

The analysis of the extreme (polar) cases yields the following intuitive facts:

(i) When the allocation procedure is perfectly inelastic, $\alpha \to 0$, the equilibrium converges to the expected situation in a *fairly lottery auction*, where each seller delivers half of the total demand. Note that, since each seller's share does not depend on the price it proposes, it will ask for the total Public Firm's budged, i.e.

$$\lim_{\alpha \to 0} p_1^*(\alpha) = \lim_{\alpha \to 0} p_2^*(\alpha) = B_0$$

(ii) On the other hand, if the allocation procedure is perfectly elastic, $\alpha \to \infty$, the equilibrium converges to the *popular*[11] equilibrium in a lower-price bought auction; i.e. if $c_1 < c_2$,

$$\lim_{\alpha \to \infty} p_1^*(\alpha) = \lim_{\alpha \to \infty} p_2^*(\alpha) = c_2; \text{ and } \lim_{\alpha \to \infty} \Psi_1^{\alpha}(B_0; P^*) = 1, \; \lim_{\alpha \to \infty} \Psi_2^{\alpha}(B_0; P^*) = 0$$

[11] The lower-price bought auction, under perfect information, has no (pure strategies) Nash equilibrium. Nevertheless, there is a tradition on predicting agents' iteration by saying that the object is sold by the lower-cost producer, and the transaltional price coincides with the second-lower cost. See Alcalde and Dahm [1] for more details.

The two facts above well might invite to think that any allocation procedure will induce a (expected) price higher than the second lowest cost. This intuition could be supported by the previous observations and the fact that p_2^*, introduced in (5.1) as a function of α, is strictly decreasing. Nevertheless, as the next theorem points out, this is not true. The reason is that equation (5.2) introduces a concave relationship between p_1^* and α. Taking into account the extreme values of p_1^* - i.e. B_0 for $\alpha \to 0$, and c_2 for $\alpha \to \infty$ - this concavity implies the existence of a value $\hat{\alpha}$ whose associated p_1^* is minimum, and thus is lower than c_2.

Theorem 5.1. *Let us assume that $c_1 < c_2 < B_0$. Then, there exists α such that, at equilibrium, $S_0(B_0, P^*) > B_0 - c_2$.*

Appendix 2 is devoted to give a formal proof of Theorem 5.1.

We can now stablish a results that points out that the nature or Corollary 3.3 can be extended to any two-provider instance.

Corollary 5.2. *Let us assume that $0 \leq c_1 < c_2 < B_0$, and let, for any given α, P^α be the unique equilibrium for Γ^α. Then there exist α^* such that*

$$U_0\left(B_0, \Psi^\alpha\left(B_0, P^\alpha\right), P^\alpha\right) > B_0 - c_2$$

6. The n-Providers Case

This section deals with the analysis of Ψ^{α} to the general n-providers case, $n \geq 3$, and how this extension can be used by the Public Firm on its own benefit.

We first introduce a natural extension of our allocation procedure to this case.

Definition 6.1. *Let us suppose that sellers' proposed prices are increasingly ordered, i.e.* $0 \leq p_1 \leq \ldots \leq p_i \leq \ldots \leq p_n \leq B_0$.[12] *For each* $\alpha > 0$, *we define the* α*-serial procedure as the function* $\Psi^{\alpha}(B_0, P)$ *described as follows*

$$\Psi_n^{\alpha}(B_0, P) = \frac{1}{n}\left(\frac{B_0 - p_n}{B_0 - p_1}\right)^{\alpha}, \qquad \textit{and}$$

$$\Psi_i^{\alpha}(B_0, P) = \frac{1}{i}\frac{(B_0 - p_i)^{\alpha} - (B_0 - p_{i+1})^{\alpha}}{(B_0 - p_1)^{\alpha}} + \Psi_{i+1}^{\alpha}(B_0, P) \quad \textit{for each } i = 1, \ldots, n-1.$$

For $\alpha = 1$ we employ the term *serial procedure*. Note that, for $n = 2$, the serial procedure coincides with Ψ^{cg}.

When analyzing the game induced by Ψ^{α} it might be possible to find the existence of a (pure strategies) Nash equilibrium with some interesting features. In particular, when agents' marginal costs are increasingly ordered, it is possible to construct a Nash equilibrium in which the prices proposed by the sellers are also increasingly ordered. Nevertheless, as examples 6.2 and 6.3 will point out, the presence of more that two sellers will introduce some 'noise' relative to what we would like to expect.

[12] *Note that, when* $p_i = B_0$ *for all seller* i *our expression is not well-defined. For such a case, we will follow the convention that* $\Psi_i^{\alpha}(B_0, P) = 1/n$ *for each agent.*

Example 6.2. *Multiplicity of Nash equilibria.*

Let us consider the following situation. $B_0 = 96$*;* $C = (78, 84, 84)$*. It is easy to check that, for* $\alpha = 1$*, the induced game has (only) two pure-strategy Nash equilibria, namely* P^**, and* $\hat{P}$*, described by*

$$P^* = (87, 89, 90)\,, \text{ and}$$
$$\hat{P} = (87, 90, 89)\,.$$

I.e., even do the symmetry of agents 2 *and* 3 *in this game, there is no symmetric (pure strategy) Nash equilibrium.*

Next example points out that, for the three-seller case, if providers' marginal cost are very close, it is possible to have some counter-intuitive Nash equilibria.

Example 6.3. *Let us consider the following three-sellers problem.* $B_0 = 6000$*;* $C = (4798, 4800, 4802)$*. Since* $c_1 < c_2 < c_3$*, what one might expect is that, at equilibrium prices should be related in a similar way, i.e.* $p_1^* < p_2^* < p_3^*$*. Nevertheles, when computing the (pure strategy) Nash equilibria, it is straightforward to see that there are three such a equilibria, namely*

$$\begin{aligned} P^* &= \left(6000 - \frac{1}{2}\sqrt{2162398}, \frac{31801}{6}, 5401\right) \approx (5264.7, 5300.2, 5401)\,; \\ \hat{P} &= \left(6000 - \sqrt{540\,299}, 5400, 5301\right) \approx (5264.9, 5400, 5301)\,; \text{ and} \\ \tilde{P} &= \left(\frac{31799}{6}, 5399, 6000 - \frac{1}{2}\sqrt{2157598}\right) \approx (5299.8, 5399, 5265.6) \end{aligned}$$

Note that $\hat{p}_1 < \hat{p}_3 < \hat{p}_2$, and $\tilde{p}_3 < \tilde{p}_1 < \tilde{p}_2$, i.e., at $\tilde{P}$ the provider proposing the lowest price is the one having the highest cost.

To sum up, let us propose some conclusions relative to the Public Firms' interest.

(a) They prefer a low budged, B_0. Note that, at equilibrium, each seller's proposed price is increasing in B_0. Nevertheless it could be exogenously given.

(b) They want to impose an α high enough to appropriately force the sellers' competition.

In what follows, we are going to consider $\alpha > 0$ to be fixed. We now deal with the possibility that a Public Firm could have to reduce the number of *active* providers and how the inactive ones (if any) would help on reducing the *relevant* budged constrain for the Public Firms. To reach our objective, we are going to consider a mechanism in which each seller will propose the price at which it is able to satisfy the demand by the Public Firm. Then, given the players' selected actions, we will share the Public Firms' needs according to Ψ^{α}, introduced in Definition 6.1. What is it new, related to the game induced by Ψ^{α} is the budged constrain to be considered. Instead of employing B_0 as a fixed parameter to compute the sellers' share, we will use an endogenous value, which depends not only on B_0 but also on the sellers' proposed prices.

Let us now to introduce some notation. Given $P = (p_1, \ldots, p_i, \ldots, p_n)$, and B_0, let

p^1 denote the lowest price, i.e., $p^1 = \min_i p_i$; and for $j > 1$, let p^j be defined as

$$p^j = \begin{cases} \min_{i \in \mathcal{I}^j} p_i & \text{if } \mathcal{I}^j = \left\{ i \in \mathcal{I} : p_i > p^{j-1} \right\} \neq \emptyset \\ B_0 & \text{otherwise} \end{cases}$$

Let us observe that p^j can be interpreted as the j-th lowest price. What we also impose is to be restricted to prices not exceeding B_0.

We can now define, for B_0 and P given, $\bar{B}$ as

(*i*) $\bar{B} = p^2$ if $\left\{ i \in \mathcal{I} : p_i = p^1 \right\}$ has at least two elements, or

(*ii*) $\bar{B} = p^3$ otherwise.

Definition 6.4. *Given the Public Firm's budged constrain, B_0, we define the α-reduced mechanism, $\Gamma^{\alpha R}$ in short, as the normal-form game $\left\{ \mathcal{I}, S, U, \Psi^{\alpha R} \right\}$ in which given agents' prices $P = (p_1, \ldots, p_i, \ldots, p_n)$, agent i's share of the Public Firm's demand is*

$$\Psi_i^{\alpha R} (B_0, P) = \Psi_i^{\alpha} \left(\bar{B}, \bar{P} \right)$$

with $\bar{p}_i = \min \left\{ p_i, \bar{B} \right\}$.

The analysis of $\Gamma^{\alpha R}$ gives some appealing properties to the equilibria. Just to introduce them, let us consider sellers' costs to be increasingly ordered, and assume that B_0 is high enough to capture the interest of seller 3, i.e. $B_0 > c_3 > c_2$. In such a case, we can establish the following:

(a) $\Gamma^{\alpha R}$ has (pure strategy) Nash equilibria;

(b) For any (pure strategy) Nash equilibrium P^*, it will holds that $p_1^* \leq p_2^*$ and $c_2 < p_2^* \leq c_3$.

(c) For each seller $i \geq 3$, and any equilibrium P^*, $U_i(B_0, P^*) = 0$.

(d) For any $\alpha > 0$, and each P^* equilibrium for $\Gamma^{\alpha R}$, if $c_1 < c_2$, and c_3 is close enough to c_2, $S_0(B_0, P^*) > B_0 - c_2$.

All the above assertions are a direct consequence of the following theorems.

Theorem 6.5. *Let assume that* $0 \leq c_1 \leq \ldots \leq c_i \leq \ldots \leq c_n \leq B_0$, *and consider* $\alpha > 0$. *Then* $\Gamma^{\alpha R}$ *has a pure strategies Nash Equilibrium* P^*. *Moreover, for each such an equilibrium it holds that*

(a) $U_i(B_0, P^*) = 0$ *for all* $i \in \mathcal{I} \setminus \{1, 2\}$; *and*

(b) $U_i(B_0, P^*) > 0$ *if, and only if* $c_i < c_3$.

Appendix 3 proposes a proof of Theorems 6.5 which illustrates how to compute the equilibria for $\Gamma^{\alpha R}$, and derives some properties satisfied at equilibrium.

Theorem 6.6. *Let us assume that sellers' marginal costs are increasingly ordered, and* $0 \leq c_1 < c_2 < c_3 < B_0$. *Then there exist* α *such that if* P^* *is an equilibrium for* $\Gamma^{\alpha R}$, *then*

$$U_0\left(B_0, \Psi^{\alpha R}\left(B_0, P^*\right), P^*\right) > S_0\left(B_0, P^*\right) > B_0 - c_2.$$

Theorem 6.6 is proved in Appendix 4.

7. Private Information and the Serial Allocation Procedure

The previous sections deal with the case where providers are perfectly informed about other's characteristics. We now focus on the case where each agent has (private) information about its own costs, but does not know which is the cost of its potential rivals.

What we propose in this section is a (continuous-time) sequential mechanism in which agents do not need such an information. I.e., if each provider knows exactly which its costs structure is, at equilibrium the price proposed by each seller coincides with the one settled at the unique equilibrium proposed in Section 4.

For α given, the mechanism proceeds as follows. At each $t \in [0,1]$, all the agents (simultaneously) select a message on $\{0,1\}$. Given agents messages, we proceed as follows:

$$\tilde{p}_i = (1 - \tilde{m}_i)\, B_0$$

where, for agent i, $\tilde{m}_i$ stands for the first time at which its action is 1, i.e. if $m_i(t)$ denotes agent i's message at t, $\tilde{m}_i = \tilde{t}$ whenever $m_i\left(\tilde{t}\right) = 1$, and $m_i(t) = 0$ for each

$t < \tilde{t}$. To complete the description of $\tilde{m}_i$, let us establish that $\tilde{m}_i = 1$ if $m_i(t) = 0$ for all t.

Then, given sellers' prices $\tilde{P} = (\tilde{p}_1, \ldots, \tilde{p}_i, \ldots, \tilde{p}_n)$, their shares on the provision are given according to $\Psi^{\alpha R}$, and thus provider i's utility is

$$\Pi_i = (\tilde{p}_i - c_i)\, \Psi_i^{\alpha R}\left(B_0; \tilde{P}\right)$$

To explore how agents play when faced to the above mechanism, we will distinghish two cases. The first one, lies with a two-provider scenario, whereas the second one is related to the situation in which there are more than two sellers.

7.1. The sequential countinous-time serial game I: The two-provider case.

To analyze the two-provider case note that, for α given, a seller i, whose opponent's message is 0 for any t lower than some $\bar{t}$, will also select, at time $\bar{t}$ a message $m_i(\bar{t}) = 0$, except if

$$(1 - \bar{t})\, B_0 = \frac{B_0 + \alpha c_i}{\alpha + 1}. \tag{7.1}$$

The reasson is that, given that its opponent has not fixed a price, by setting a message equal to 1, it is the seller asking for the highest price. As we have seen in Section 4, the provider i, when asking for the highest price, selects p_i as established in equation (7.1).

Now, for $\tilde{t}_j$ given, let us assume that seller j sets a message $m_j(t) = 0$ for $t < \tilde{t}_j$, and $m_j\left(\tilde{t}_j\right)=1$. Note that, if $\tilde{t}_j$ satisfies

$$\tilde{t}_j<\frac{\alpha}{\alpha+1}\frac{B_0-c_i}{B_0}$$

then, provider $i\neq j$ best option is to choose messages m_i such that $m_i\left(t\right)=0$ for $t<\tilde{t}_i$, and $m_i\left(\tilde{t}_i\right)=1$, where $\tilde{t}_i$ is the unique solution to[13]

$$\max\left[\left(1-t_i\right)B_0-c_i\right]\Psi_1^{\alpha}\left(1;1-t_i,1-\tilde{t}_j\right)\text{ s.t. }1\geq t_i\geq\tilde{t}_j.$$

Therefore, following the above arguments, we can establish the next theorem, whose proof is ommited, establishing the *equivalence* of the shares proposed, at equilibrium, under Γ^{α} and $\Gamma^{c\alpha}$.

For simplicity of exposition, we denote by $\Gamma^{c\alpha}$ the two-player game in which agents select a message for each $t\in[0,1]$; at time t each agent knows its opponent's messages for any $t'<t$; and their share is established according Ψ^{α} for agents' prices determined by $\tilde{p}_i=\left(1-\tilde{m}_i\right)B_0$ as previously explained. An strategy for provider i, s_i, is a function that determines, at each t, a message which deppends on the messages settled for any $t'<t$.

[13] Note that, given Ψ^{α} is both anonymous and homogeneous, the expression

$$\Psi_1^{\alpha}\left(1;1-t_i,1-\tilde{t}_j\right)$$

coincides with i's share when its price is $\left(1-t\right)B_0$, and the settled by its rival is $\left(1-\tilde{t}_j\right)B_0$, provided that the Public Firm's budged is B_0.

Theorem 7.1. *Let us assume that $c_1 \leq c_2 < B_0$. Given (s_1^*, s_2^*), a profile of strategies for $\Gamma^{c\alpha}$, let m_i^* denote the seller i's message function induced by (s_1^*, s_2^*). Then, (s_1^*, s_2^*) is an equilibrium for $\Gamma^{c\alpha}$ if, and only if,*

(i) $m_2^*(t) = 0$ *for each* $t < t_2$*, and* $m_2^*(t_2) = 1$*; and*

(ii) $m_1^*(t) = 0$ *for each* $t < t_1$*, and* $m_1^*(t_1) = 1$*, where*

$$\begin{aligned} t_2 &= \frac{\alpha}{\alpha+1}\frac{B_0 - c_2}{B_0}, \text{ and} \\ & t_1 \text{ maximices } \left[(1-t)B_0 - c_1\right]\Psi_1^{\alpha}\left(B_0; (1-t_1)B_0, (1-t_2)B_0\right). \end{aligned}$$

Note that a consequence of Theorem 7.1 we have the following. Let us consider a two-agent instance, with $c_1 \leq c_2 < B_0$. Let (p_1^*, p_2^*) be the equilibrium prices when agents play the game Γ^{α}, and $(\hat{p}_1, \hat{p}_2)$ those established when playing $\Gamma^{c\alpha}$. Then, $(p_1^*, p_2^*) = (\hat{p}_1, \hat{p}_2)$.

7.2. The sequential countinous-time serial game II: The n-provider case.

Let us consider now the n-provider case, $n > 2$. Since agents have only information about their own cost, each agent could guess that, at any t, all its opponents might had selected a message $m_j(t) = 1$. Therefore, we should expect that any seller will select a message ensuring a non-negative utility. Thus, for each i there should be a t such that

$m_i(t) = 1$ and[14]

$$t \leq \frac{B_0 - c_i}{B_0} = t_i. \tag{7.2}$$

To continue exploring the providers' attitude when faced to this mechanism, let us introduce some additional notation. Given $t \in [0, 1]$, let $O(t)$ denote the set of sellers that already have stablished their argued price at t, i.e.

$$O(t) = \left\{ i \in \mathcal{I} : m_i\left(t'\right) = 1 \text{ for some } t' < t \right\},$$

and let $o(t)$ denote its cardinal, i.e. the number of providers in $O(t)$. Finally, given the sellers' messages, let t^k denote the k-th lowest value of t at which a provider fix its selected price, i.e. t^k is such that $o(t) < k$ for each $t < t^k$, and $o(t) \geq k$ for any $t > t^k$.

Note that at time t, provider i being not in $O(t)$ will select $m_i(t) = 0$ whenever $o(t) \leq n - 3$ and $(1 - t) B_0 > c_i$. The reason is that it could improve its profits by waiting having no risk of incurring on losses.

The above can be summarized in the following theorem, whose proof is ommited.

Theorem 7.2. *Let us assume that* $c_1 \leq c_2 \leq \ldots \leq c_i \leq \ldots \leq c_n < B_0$. *Given* $s^* = (s_1^*, \ldots, s_i^*, \ldots, s_n^*)$, *a profile of strategies for* $\Gamma^{c\alpha R}$, *let* m_i^* *denote the seller* i*'s*

[14]Note that if the lower t at which a seller sets a message $m_i(t) = 1$ is

$$t = \frac{B_0 - c_i}{B_0},$$

then the price it is asking for coincides to its marginal cost c_i. This bound is benoted by t_i.

message function induced by s^*. *Then,* s^* *is an equilibrium for* $\Gamma^{c\alpha R}$ *if, and only if,*

(i) *For each* $i > 2$, $m_i^*(t) = 0$ *for each* $t < t_i$, *and* $m_i^*(t_i) = 1$;

(ii) $m_2^*(t) = 0$ *for each* $t < t_2$, *and* $m_2^*(t_2) = 1$; *and*

(iii) $m_1^*(t) = 0$ *for each* $t < t_1$, *and* $m_1^*(t_1) = 1$, *where*

$$t_i = \frac{B_0 - c_i}{B_0} \text{ for each } i > 2;$$

$$t_2 = \frac{(\alpha+1)\,B_0 - c_3 - \alpha c_2}{(\alpha+1)\,B_0}; \text{ and}$$

$$t_1 \text{ maximices } [(1-t)\,B_0 - c_1]\,\Psi_1^{\alpha R}(1; (1-t_1), \ldots, (1-t_i), \ldots, (1-t_n)).$$

Note that a consequence of Theorem 7.2 we have the following. Let us consider a n-agent instance, with $c_i \leq c_{i+1}$ for each $i < n$, and $c_n < B_0$. Let $P^* = (p_1^*, \ldots, p_n^*)$ be the equilibrium prices, when agents play the game $\Gamma^{\alpha R}$, in which $p_i^* = c_i$ for all $i > 2$, and $\hat{P} = (\hat{p}_1, \ldots, \hat{p}_n)$ those established when playing $\Gamma^{c\alpha}$. Then, $P^* = \hat{P}$.

8. Concluding Comments

In this paper we proposed a new procedure to solve the problem of how much a Public Firm should buy to each potential seller. In this procedure there are some interesting features that are particularly remarkable:

1. At least two sellers have a true incentive to participate in the allocation procedure. This is because they will have, for sure, a positive benefit.

2. The Public Firm is somehow safe against some provider's failure because there are, at least, another active provider that might cover any potential underproduction by its opponent.

3. The Public Firm could induce an expenditure lower than the one expected in the classical auctions. To reach its objective, it might select a procedure in which the price-elasticity of the sharing rule is high enough.

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APPENDIX 1. ***The Analysis of*** $\Gamma^{\alpha d}$. ***Proving theorems 4.2 and 4.3***.

Throughout this appendix we will consider fixed parameters satisfying

$$\alpha > 0;\text{ and } 0 \leq c_1 \leq c_2 < E.$$

We will proceed as follows. We first will show, Lemma A.1.1, that the allocation procedure function is continuously differentiable. Secondly, throughout Lemma A.1.2, we will see that each seller's utility function is strictly cuasi-concave on the bidding space. Finally, we will use a result by Moulin and Shenker [8] to build an equilibrium and to show that this equilibrium is unique.

Once Theorem 4.2 has been proven, we will deal with a proof for Theorem 4.3.

Lemma A.1.1. *Let k such that $0 < k < B_0$. The function $f : [0, B_0] \to \mathbb{R}_+$ described by*

$$f(x) = \begin{cases} \frac{1}{2}\left(\frac{B_0 - x}{B_0 - k}\right)^{\alpha} & \text{if } k < x \leq B_0 \\ \frac{2(B_0-x)^{\alpha} - (B_0-k)^{\alpha}}{2(B_0-x)^{\alpha}} & \text{if } 0 \leq x \leq k \end{cases}$$

is continuously differentiable in $(0, B_0)$.

Proof. Note that, since f is polynomial for $x \neq k$, we only need to check that

$$\lim_{x_0 \to k^-} \frac{\partial f}{\partial x}(x_0) = \lim_{x_0 \to k^+} \frac{\partial f}{\partial x}(x_0)$$

Now, taking into consideration that

$$\frac{\partial f}{\partial x}(x_0) = \begin{cases} \frac{\alpha}{2}\frac{(B_0-x_0)^{\alpha-1}}{(B_0-k)^{\alpha}} & \text{if } k < x_0 < B_0 \\ \frac{\alpha}{2}\frac{(B_0-k)^{\alpha}}{(B_0-x_0)^{\alpha+1}} & \text{if } 0 < x_0 < k \end{cases}$$

we conclude that

$$\begin{aligned} \lim_{x_0 \to k^-} \frac{\partial f}{\partial x}(x_0) &= \frac{\alpha}{2}\frac{(B_0-k)^{\alpha}}{(B_0-k)^{\alpha+1}} = \frac{\alpha}{2(B_0-k)}, \text{ and} \\ \lim_{x_0 \to k^+} \frac{\partial f}{\partial x}(x_0) &= \frac{\alpha}{2}\frac{(B_0-k)^{\alpha-1}}{(B_0-k)^{\alpha}} = \frac{\alpha}{2}(B_0-k)^{-1} \end{aligned}$$

■

Lemma A.1. 2. *For each* k, $0 < k < B_0$ *and any* h *in* $[0, B_0]$ *the function* $g : [0, B_0] \to \mathbb{R}$ *described by*

$$g(x) = \begin{cases} \frac{1}{2}\left(\frac{B_0-x}{B_0-k}\right)^{\alpha}(x-h) & \text{if } k < x \leq B_0 \\ \frac{2(B_0-x)^{\alpha}-(B_0-k)^{\alpha}}{2(B_0-x)^{\alpha}}(x-h) & \text{if } 0 \leq x \leq k \end{cases}$$

is strictly cuasi-concave in (h, B_0).

Proof. Note that, given h, if f is the function explored in Lemma A.1.1, g can be described as

$$g(x) = (x-h)f(x)$$

which is twice continuous differentiable for $x \neq k$, and it is continuously differentiable for $x = k$. Let us consider the following cases:

(a) $h < x_0 < k$. Then

$$\frac{\partial^2 g}{\partial x^2}(x_0) = \frac{1}{2}\alpha \frac{(B_0 - k)^\alpha}{(B_0 - x_0)^2 (B_0 - x_0)^\alpha}(h + x_0 - 2B_0 + h\alpha - x_0\alpha)$$

Therefore, $sign\ \frac{\partial^2 g}{\partial x^2}(x_0) = sign\,(h + x_0 - 2B_0 + h\alpha - x_0\alpha)$. Since $h \leq x_0 < B_0$,

$$h + x_0 - 2B_0 + h\alpha - x_0\alpha = (1+\alpha)\,h + (1-\alpha)\,x_0 - 2B_0 \leq 2\,(x_0 - B_0) < 0$$

(b) $k < x_0 < B_0$. Then

$$\frac{\partial^2 g}{\partial x^2}(x_0) = \frac{1}{2}\frac{\alpha}{(B_0 - x_0)^2}\left(\frac{B_0 - x_0}{B_0 - k}\right)^\alpha (h + x_0 - 2B_0 - \alpha h + \alpha x_0)$$

So, given that $sign\ \frac{\partial^2 g}{\partial x^2}(x_0) = sign\,(h + x_0 - 2B_0 - h\alpha + \alpha x_0)$, we have that $\frac{\partial^2 g}{\partial x^2}(x_0) \geq 0$ only if

$$h - B_0 \geq B_0 - x_0 + \alpha\,(h - x_0)\,.$$

But, since

$$\frac{\partial g}{\partial x}(x_0) = \frac{(B_0 - x_0)^\alpha}{2(B_0 - k)^\alpha}\left(1 - \alpha\frac{x_0 - h}{B_0 - x_0}\right) = \frac{(B_0 - x)^\alpha}{2(B_0 - k)^\alpha}\left(\frac{B_0 - x + \alpha(h - x)}{B_0 - x}\right),$$

we have that, g is decreasing at x_0 if it is not strictly concave.

Summing up, we have that g is strictly concave at x_0 whenever

$$x_0 < \frac{2B_0 + (\alpha - 1)h}{\alpha + 1},$$

and for x_0 higher or equal the above, if g is not strictly concave, it is strictly decreasing. Thus g must be strictly cuasi-concave. ■

Now, let us introduce the following lemma, which is equivalent to Lemma 2 in Moulin and Shenker [8].

Lemma A.1. 3. *Let $u_1(x)$ and $u_2(x)$ be two strictly cuasi-concave functions from $[a, b]$ to $\mathbb{R}$ that coincide from $c \in (a, b)$:*

$$u_1(x) = u_2(x) \text{ for all } x,\ c \le x \le b.$$

Then the (unique) maximizers of $u_i(x)$, denoted x_i, are on the same side of c:

$$x_1 \ge c \Leftrightarrow x_2 \ge c;\ x_1 = c \Leftrightarrow x_2 = c$$

We can now proceed to prove Theorem 4.2.

Proof of Theorem 4.2

Let us assume that $0 \leq c_1 \leq c_2 < B_0$. We first show that a (pure strategies) equilibrium exists. Let p_1^* and p_2^* be the solutions to

$$\begin{aligned} \max U_2\left(B_0; p_1^*, p_2\right) \; s.t. \; p_1^* &\leq p_2 \leq B_0, \text{ and} \\ \max U_1\left(B_0; p_1, p_2^*\right) \; s.t. \; c_1 &\leq p_1 \leq p_2^*. \end{aligned}$$

Note that the sellers' strategies can be computed using the following sequential procedure:

(a) Calculate p_2^* by maximizing, on $[c_2, B_0]$, the function

$$m_2\left(p_2\right) = \left(B_0 - p_2\right)^{\alpha}\left(p_2 - c_2\right)$$

Note that this function is strictly cuasi-concave on this segment. Moreover, any solution of the problem above also maximizes the function

$$\frac{\left(B_0 - p_2\right)^{\alpha}}{2\left(B_0 - \hat{p}_1\right)^{\alpha}}\left(p_2 - c_2\right)$$

which coincides with $U_2\left(B_0; \hat{p}_1, p_2\right)$ when $\hat{p}_1 \leq p_2$.

(b) Compute p_1^* by maximizing, on $[c_1, p_2^*]$, the function

$$m_1(p_1) = \frac{2(B_0 - p_1)^\alpha - (B_0 - p_2^*)^\alpha}{2(B_0 - p_1)^\alpha}(p_1 - c_1).$$

Note that, since m_1 is strictly cuasi-concave, it must have a unique maximizer.

Let us remark that (p_1^*, p_2^*) is characterized by

$$\begin{aligned} p_2^* &= \frac{B_0 + \alpha c_2}{\alpha + 1}, \\ p_1^* \text{ satisfies } (B_0 - p_1^*) &= (B_0 - \alpha c_1 + (\alpha - 1)p_1^*)\frac{(B_0 - p_2^*)^\alpha}{2(B_0 - p_1^*)^\alpha} \end{aligned}$$

Now we will show that (p_1^*, p_2^*) described above constitutes an equilibrium for the game induced by Ψ^α. First, note that $\hat{p}_1$, the unique maximizer of

$$u_1(p_1) = \frac{(B_0 - p_1)^\alpha}{2(B_0 - k)^\alpha}(p_1 - c_1)$$

is, for any $k < B_0$,

$$\hat{p}_1 = \frac{B_0 + \alpha c_1}{\alpha + 1} \le p_2^* \tag{8.1}$$

Therefore, since for $k = p_2^*$, the function u_1 and seller 1's utility

$$U_1(B_0; p_1, p_2^*) = \begin{cases} \frac{(B_0 - p_1)^\alpha}{2\left(B_0 - p_2^*\right)^\alpha}(p_1 - c_1) & \text{if } p_2^* < p_1 \le B_0 \\ \frac{2(B_0 - p_1)^\alpha - \left(B_0 - p_2^*\right)^\alpha}{2(B_0 - p_1)^\alpha}(p_1 - c_1) & \text{if } 0 \le p_1 \le p_2^* \end{cases}$$

coincide on $[p_2^*, B_0]$, by Lemma A.1.3 we have that p_1^* is the seller 1's best-response to agent 2's strategy. Applying a similar reasoning to agent 2 we see that p_2^* is its best-response to p_1^*.

We now deal with the equilibrium uniqueness. On the contrary, let us assume that there is an equilibrium $(\tilde{p}_1, \tilde{p}_2) \neq (p_1^*, p_2^*)$. Note that, by the above arguments, we can ensure that $\tilde{p}_1 > \tilde{p}_2$. Otherwise, we could argue that $(\tilde{p}_1, \tilde{p}_2)$ is not an equilibrium.

Now, since $\tilde{p}_1 > \tilde{p}_2$, agent 1's strategy should be a maximizer of

$$\frac{(B_0 - p_1)^\alpha}{2\,(B_0 - \tilde{p}_2)^\alpha}\,(p_1 - c_1)\,,$$

i.e. $\tilde{p}_1$ should follow expression (8.1) above. Since the functions

$$u_2\,(p_2) = \frac{(B_0 - p_2)^\alpha}{2\,(B_0 - \tilde{p}_1)^\alpha}\,(p_2 - c_2)\,, \text{ and}$$

$$U_2\,(B_0; \tilde{p}_1, p_2) = \begin{cases} \frac{(B_0 - p_2)^\alpha}{2(B_0 - \tilde{p}_1)^\alpha}\,(p_2 - c_2) & \text{if } \tilde{p}_1 < p_2 \leq B_0 \\ \frac{2(B_0 - p_2)^\alpha - (B_0 - \tilde{p}_1)^\alpha}{2(B_0 - p_2)^\alpha}\,(p_2 - c_2) & \text{if } 0 \leq p_2 \leq \tilde{p}_1 \end{cases}$$

coincide on $[\tilde{p}_1, B_0]$, and $p_2^* \geq \tilde{p}_1$ is the unique maximizer of u_2, by Lemma A.1.3 we know that $\tilde{p}_2 \geq \tilde{p}_1$. A contradiction. ■

Proof of Theorem 4.3

First, note that when

$$d \leq \frac{B_0 + \alpha \max\{c_1, c_2\}}{1+\alpha} \tag{8.2}$$

the proof is straightforward. This is because d is a dominant strategy for provider i satisfying

$$d \leq \frac{B_0 + \alpha c_i}{1+\alpha}.$$

Therefore, we can conclude that

(a) If $(1+\alpha)\, d \leq B_0 + \alpha \min\{c_1, c_2\}$, both agents have a donimant strategy, d, and the result is proven; or

(b) If there is a unique agent, say 2, such that $(1+\alpha)\, d \leq B_0 + \alpha c_2$. In such a case, provider 2 has a dominant strategy, which is d, and therefore, agent 1 will have a unique undominated strategy, given that agent 2 plays d. This strategy is the unique maximizer of $U_i\,(B_0; p_1, d)$.

Therefore, we will assume in the rest of the proof that

$$\frac{B_0 + \alpha \max\{c_1, c_2\}}{1+\alpha} < d < B_0.$$

We will proceed as follows. We will propose an iterative procedure to sequentially reduce each seller's set of undominated strategies. We next will see that this process is convergent and that the limit set of (sequentially) undominated strategies is a singleton.

First, note that it is straightforward to see that, for each agent the set of undominated strategies is (a subset of)

$$S_i^0 = (c_i, d]$$

The reason is that an agent selecting a strategy outside S_i^0 will obtain a non-positive utility, whereas any strategy inside S_i^0 will report a positive utility to that agent.

Now, consider the intervals $S_i^1 = \left[\boldsymbol{l}_i^1, \boldsymbol{u}_i^1\right]$, where

$$\boldsymbol{u}_i^1 = \arg\max U_i\left(B_0, (p_i, d)\right) \tag{8.3}$$

and $\boldsymbol{l}_i^1$ is the minimal price on S_i^0 such that[15]

$$U_i\left(B_0, \left(\boldsymbol{u}_i^1, c_j\right)\right) \geq U_i\left(B_0, (p_i, c_j)\right). \tag{8.4}$$

Note that, since U_i is continuous, strictly quasi-concave (for p_j given), and also $U_i\left(B_0, \left(\boldsymbol{u}_i^1, c_j\right)\right) > 0$, we have the existence of $\boldsymbol{l}_i^1 > c_i$ being the minimal solution for inequation [8.4].

Let us observe that $\boldsymbol{u}_i^1$ dominates any strategy outside S_i^1 when the other seller is selecting an strategy on S_j^0. The reason is the following. Let us assume that provider j selects p_j, and consider the following cases:

[15] As usual, when we are analyzing provider i, sub-index j will refer the other agent.

Case 1. $p_j \leq \frac{B_0+\alpha c_i}{1+\alpha}$. Note that, in such a case, the best strategy for provider i is $\hat{p}_i = \frac{B_0+\alpha c_i}{1+\alpha}$, and its utility function, for p_j fixed, is strictly increasing on $\left[c_i, \frac{B_0+\alpha c_i}{1+\alpha}\right]$. Therefore, for each $p_i < \boldsymbol{l}_i^1$

$$U_i\left(B_0, (p_i, p_j)\right) < U_i\left(B_0, \left(\boldsymbol{l}_i^1, p_j\right)\right) = U_i\left(B_0, \left(\boldsymbol{u}_i^1, p_j\right)\right)$$

Case 2. $p_j > \frac{B_0+\alpha c_i}{1+\alpha}$. Note that, in such a case, the best reply for seller i is a price $\hat{p}_i$ that belongs to $\left(\frac{B_0+\alpha c_i}{1+\alpha}, \boldsymbol{u}_i^1\right]$. Moreover, for p_j given, i's utility function is decreasing on $[p_j, d]$. Therefore, for each $p_i > \boldsymbol{u}_i^1$

$$U_i\left(B_0, \left(\boldsymbol{u}_i^1, p_j\right)\right) > U_i\left(B_0, (p_i, p_j)\right).$$

So, $\boldsymbol{u}_i^1$ dominates any strategy outside S_i^1.

Now, let us construct, for each provider i, the sequence $S_i^k = \left[\boldsymbol{l}_i^k, \boldsymbol{u}_i^k\right]$, where

$$\boldsymbol{u}_i^k = \arg\max U_i\left(B_0, \left(p_i, \boldsymbol{u}_j^{k-1}\right)\right) \tag{8.5}$$

and $\boldsymbol{l}_i^k$ is the minimal price on S_i^{k-1} such that $U_i\left(B_0, \left(\boldsymbol{u}_i^1, \boldsymbol{u}_j^{k-1}\right)\right) \geq U_i\left(B_0, \left(p_i, \boldsymbol{u}_j^{k-1}\right)\right)$.

Let us observe that, for each agent i, and iteration k, $S_i^{k+1} \subseteq S_i^k$, and this inclusion is strict whenever $\boldsymbol{u}_i^k > \frac{B_0+\alpha c_i}{1+\alpha}$. The reason is that each agent utility function is strictly quasi-concave, and that $\boldsymbol{u}_i^k$ is strictly increasing in $\boldsymbol{u}_j^{k-1}$, which is decreasing on k

whenever S_j^k is not a singleton.

Therefore, the sequence of intervals $\left\{S_i^k\right\}_{k=1}^{\infty}$ is decreasing (in length), and thus convergent.

Moreover, since

$$\left.\frac{d\boldsymbol{u}_i}{d\boldsymbol{u}_j}\right|_{\boldsymbol{u}_i<\boldsymbol{u}_j} = -\frac{\frac{\partial\left(1-\frac{1}{2}(B_0-\boldsymbol{u}_j)^{\alpha}\left(\frac{B-\boldsymbol{u}_i+\alpha(\boldsymbol{u}_i-c_i)}{(B_0-\boldsymbol{u}_i)^{\alpha+1}}\right)\right)}{\partial\boldsymbol{u}_j}}{\frac{\partial\left(1-\frac{1}{2}(B_0-\boldsymbol{u}_j)^{\alpha}\left(\frac{B-\boldsymbol{u}_i+\alpha(\boldsymbol{u}_i-c_i)}{(B_0-\boldsymbol{u}_i)^{\alpha+1}}\right)\right)}{\partial\boldsymbol{u}_i}} =$$

$$= \frac{(B_0-\boldsymbol{u}_i)}{(B_0-\boldsymbol{u}_j)}\frac{B_0-\boldsymbol{u}_i+(\boldsymbol{u}_i-c_i)\,\alpha}{B_0-\boldsymbol{u}_i+(\boldsymbol{u}_i-c_i)\,\alpha+B_0-c_i} > \frac{(B_0-\boldsymbol{u}_i)}{(B_0-\boldsymbol{u}_i)+(B_0-c_i)} > \frac{1}{2}$$

we have that, whenever $\boldsymbol{u}_i^k < \boldsymbol{u}_j^{k-1}$, $\boldsymbol{u}_i^k - \boldsymbol{u}_i^{k+1}$ is always positive and does not converge to zero. This implies that the succession $\left\{S_i^k\right\}_{k=1}^{\infty}$ converges to a singleton. ■

APPENDIX 2.

This appendix is devoted to give a formal proof for Theorem 5.1 and Corollary 5.2. Note Theorem 3.2 and Corollary 3.3 are a direct consequence of the previous results for $\alpha = 1$. For the sake of simplicity, we will concentrate on a family $\mathcal{F}$ of functions f_α such that $f_\alpha(x) = x^\alpha$, with $\alpha > 0$. Therefore, the allocation procedure induced by a function f_α is

$$\Psi^\alpha(B_0, P) = \begin{cases} \left(\frac{1}{2}\left(\frac{B_0 - p_1}{B_0 - p_2}\right)^\alpha, 1 - \frac{1}{2}\left(\frac{B_0 - p_1}{B_0 - p_2}\right)^\alpha\right) & \text{if } p_2 \leq p_1 \leq B_0 \\ \\ \left(1 - \frac{1}{2}\left(\frac{B_0 - p_2}{B_0 - p_1}\right)^\alpha, \frac{1}{2}\left(\frac{B_0 - p_2}{B_0 - p_1}\right)^\alpha\right) & \text{if } p_1 < p_2 \leq B_0 \end{cases}$$

Let us observe that when $0 \leq c_1 < c_2 < B_0$, at equilibrium,[16] $P^* = (p_1^*, p_2^*)$ must satisfy

$$p_2^* = \frac{B_0 + \alpha c_2}{\alpha + 1}, \tag{8.6}$$

and

$$(B_0 - p_1^*) = \frac{1}{2}\left(\frac{B_0 - p_2^*}{B_0 - p_1^*}\right)^\alpha [B_0 + (\alpha - 1)p_1^* - \alpha c_1] \tag{8.7}$$

By equation (8.6) we know that, by selecting α high enough, we can guarantee that p_2^* is very close to c_2. Moreover, as Lemma A.2.1 following establishes, if $c_1 < c_2$, we can select α such that $p_1^* < c_2$. The conclusion by Theorem 5.1 is a direct consequence

[16] Note that P^* will deppend on α. Therefore, we will abuse notation by using P^* instead $P^*(\alpha)$.

of the above facts.

Lemma A.2. 1. *Let assume that* $0 \leq c_1 < c_2 < B_0$*. Then there is* $\hat{\alpha} > 1$ *such that, at equilibrium*

$$p_1^* = \frac{\hat{\alpha}}{\hat{\alpha}-1}c_2 - \frac{1}{\hat{\alpha}-1}B_0.$$

Proof. Let us construct the function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(\alpha) = \frac{\alpha}{\alpha-1} - \frac{\alpha(c_2-c_1)}{2(B_0-c_2)}\left(\frac{\alpha-1}{\alpha+1}\right)^{\alpha}$$

Let us observe that g is continuos for $\alpha > 1$. Moreover,

(i) $\lim_{\alpha \to 1^+} g(\alpha) = +\infty$, and

(ii) $\lim_{\alpha \to +\infty} g(\alpha) = -\infty$.

Thus, by applying Bolzano's Theorem, we get the existence of $\hat{\alpha}$ such that $g(\hat{\alpha}) = 0$. Taking into account that g is strictly decreasing, we also can guarantee that such an $\hat{\alpha}$ is unique.

Just to conclude, let us observe that $g(\hat{\alpha}) = 0$ implies that

$$p_1^* = \frac{\hat{\alpha}}{\hat{\alpha}-1}c_2 - \frac{1}{\hat{\alpha}-1}B_0$$

is a solution of

$$(B_0 - p_1^*) = \frac{1}{2}\left(\frac{B_0 - p_2^*}{B_0 - p_1^*}\right)^{\hat{\alpha}} [B_0 + (\hat{\alpha} - 1)\, p_1^* - \hat{\alpha} c_1]$$

which is the desiderated result. ■

We next show that, by selecting the parameter α appropriately, the Public Firm can guarantee that its saving exceeds $B_0 - c_2$. Let us observe that the statement of Theorem 5.1 is a direct consequence of our next result.

Lemma A.2. 2. *Let suppose that* $0 \leq c_1 < c_2 < B_0$. *Then there exist* $\tilde{\alpha}$ *such that, at equilibrium,* $S_0\,(B_0, P^*) > B_0 - c_2$.

Proof. Given sellers' costs, let $\tilde{\alpha} > 1$ be such that

$$B_0 - p_1^* = \frac{\tilde{\alpha}}{\tilde{\alpha} - 1}\,(B_0 - c_2)$$

whose existence was proved in Lemma A.2.1 above. Let us observe that, in such a case,

$$S_0\,(B_0, P^*) = \frac{1}{2}\left(\frac{B_0 - p_2^*}{B_0 - p_1^*}\right)^{\tilde{\alpha}} (B_0 - p_2^*) + \left[1 - \frac{1}{2}\left(\frac{B_0 - p_2^*}{B_0 - p_1^*}\right)^{\tilde{\alpha}}\right] (B_0 - p_1^*)$$

Taking into account that

$$B_0 - p_1^* = \tfrac{\tilde{\alpha}}{\tilde{\alpha}-1}\left(B_0 - c_2\right), \text{ and}$$

$$B_0 - p_2^* = \tfrac{\tilde{\alpha}}{\tilde{\alpha}+1}\left(B_0 - c_2\right),$$

we have that

$$\frac{B_0 - p_2^*}{B_0 - p_1^*} = \left(\frac{\frac{\tilde{\alpha}}{\tilde{\alpha}+1}}{\frac{\tilde{\alpha}}{\tilde{\alpha}-1}}\right) = \frac{\tilde{\alpha}-1}{\tilde{\alpha}+1} < 1,$$

and thus

$$\Psi_2^{\tilde{\alpha}}\left(B_0, P^*\right) < \frac{1}{2}, \text{ and } \Psi_1^{\tilde{\alpha}}\left(B_0, P^*\right) > \frac{1}{2}.$$

Therefore,

$$S_0\left(B_0, P^*\right) > \frac{1}{2}\left[\left(B_0 - p_1^*\right) + \left(B_0 - p_2^*\right)\right] = \frac{\tilde{\alpha}^2}{\tilde{\alpha}^2 - 1}\left(B_0 - c_2\right) > \left(B_0 - c_2\right).$$

Which is the desired result. ■

To prove Corollary 5.2 if is useful to introduce the following proposition whose straightforward proof is omitted.

Proposition A.2 1. *Let μ and b two paramenters such that $0 < \mu < 1$, and $b > 0$. Then, for any $a > 0$,*

$$\left(a^{\mu} + b^{\mu}\right)^{\frac{1}{\mu}} > a + b$$

Now we can proceed to prove Corollary 5.2.

Proof of Corollary 5.2

Since the Public Firm's utility can be represented by a CES function

$$U_0\left(B_0,\pi,P\right)=\left[\sum_{i=1}^{n}\left(\pi_i\left(B_0-p_i\right)\right)^{\rho}\right]^{\frac{1}{\rho}},\text{ where }0<\rho<1$$

we can apply Proposition A.2.1 as follows

$$U_0\left(B_0,\pi,P\right)=\left[\sum_{i=1}^{n}\left(\pi_i\left(B_0-p_i\right)\right)^{\rho}\right]^{\frac{1}{\rho}}>\sum_{i=1}^{n}\pi_i\left(B_0-p_i\right)=S_0\left(B_0,\pi,P\right)$$

Now, by applying Theorem 5.1 we get the reached result. ■

APPENDIX 3. *A Proof of Theorem 6.5.*

To explore the α-reduced mechanism we will consider throughout this appendix that all the parameters are fixed and they satisfy that $\alpha > 0$, and $0 \leq c_1 \leq \ldots \leq c_i \leq \ldots \leq c_n \leq B_0$.

We will explore several cases, relative to how the marginal costs of the sellers are related. The statement established in Theorem 6.5 is a direct consequence of the study provided in this appendix. Since we are interested in the analysis of pure strategies Nash equilibria we will omit throughout this appendix a (reiterative) mention of the fact that we will not consider mixed strategies Nash equilibria.

Lemma A.3. 1. *Let assume that $c_1 \leq c_2 < c_3$. Then each P^* such that*

$$p_i^* \geq \bar{B} \text{ for all } i \in \mathcal{I} \setminus \{1,2\}\,; \tag{8.8}$$
$$p_2^* = \frac{\bar{B} + \alpha c_2}{\alpha + 1} \leq c_3, \text{ and}$$
$$p_1^* \text{ satisfies } \left(\bar{B} - p_1^*\right) = \left(\bar{B} - \alpha c_1 + (\alpha - 1)\, p_1^*\right) \frac{\left(\bar{B} - p_2^*\right)^\alpha}{2\left(\bar{B} - p_1^*\right)^\alpha}$$

constitutes an equilibrium for $\Gamma^{\alpha R}$. Moreover, each equilibrium for $\Gamma^{\alpha R}$ satisfies condition (8.8)*.*

Proof. We first prove that the strategies profiles proposed in Lemma A.3.1 constitute a Nash equilibrium for $\Gamma^{\alpha R}$. Let us consider P^* satisfying condition (8.8) above. Since

$p_2^* \leq c_3$ agents 3 to n have no interest on selecting a strategy different from the proposed one. This is because if any of such agents, say i, changes its selected price to $p_i^{'}$ it could change the share of the Public Firm's demand only if $p_i^{'} < c_i$ and thus its utility will be negative.

Now, given that agents other than 1 and 2 have no interest on selecting a different price, by Theorem 4.2 we have that agents' 1 and 2 are playing their optimal replies to the others' strategies.

On the other hand, let us consider that $\tilde{P} = (\tilde{p}_1, \ldots, \tilde{p}_i, \ldots, \tilde{p}_n)$ is an equilibrium. Without loss of generality, let suppose that $\tilde{p}_1 \leq \tilde{p}_2$. Let us observe that if $\tilde{p}_2 < \tilde{p}_i$ for all $i > 2$, then $\tilde{p}_2 \leq c_3$. Note that otherwise agent 3 could propose

$$p_3^{'} = \frac{\tilde{p}_2 + \alpha c_3}{\alpha + 1}$$

that satisfies $c_3 < p_3^{'} < \tilde{p}_2$ guaranteeing itself to reach a positive utility.

Now let us suppose that there is $i \notin \{1, 2\}$ such that

$$\tilde{p}_i = \min_{k \in \mathcal{I} \setminus \{1,2\}} \tilde{p}_k \leq \tilde{p}_2.$$

This implies that $\Psi_i^{\alpha R}\left(B_0, \tilde{P}\right) > 0$. Since $\tilde{P}$ is an equilibrium, $\tilde{p}_i > c_i$. It is easy to see that, in such a case,

$$p_2^{'} = \frac{\tilde{p}_i + \alpha c_2}{\alpha + 1}$$

would improve agent 2's utility.

Therefore we have that $\tilde{P}$ must satisfy that

$$\begin{aligned} \max\{\tilde{p}_1, \tilde{p}_2\} &< \min_{i \in \mathcal{I} \setminus \{1,2\}} \tilde{p}_i, \text{ and} \\ \max\{\tilde{p}_1, \tilde{p}_2\} &\leq c_3 \end{aligned}$$

So, by applying Theorem 4.2 we reach the desiderated result ■

We now deal with the case in which $c_1 < c_2 = c_3$. As we will see, at equilibrium, the only agent having a positive utility will be seller 1. Moreover there will be two or more agents with a positive share whose utility will be 0.

Lemma A.3. 2. *Let assume that $c_1 < c_2 = c_3$. Then $\bar{P}$ is an equilibrium if, and only if,*

- (a) $\bar{p}_i > c_2$ *for each i such that $c_i > c_2$;*
- (b) $\bar{p}_i \geq c_2$ *for each i such that $c_i = c_2$, and there are at least two sellers, i and j, such that $\bar{p}_i = \bar{p}_j = c_2$; and*
- (c) $\bar{p}_1$ *maximizes $\left(1 - \frac{k}{k+1}\left(\frac{p^3 - c_2}{p^3 - p_1}\right)^{\alpha}\right)(p_1 - c_1)$ on $[0, c_2]$, where k denotes the number of agents such that $\bar{p}_l = c_2$.*

Proof. We first prove that the strategies profiles proposed in Lemma A.3.2 constitute an equilibrium. Note that no agent other than 1 has any interest on playing a strategy different from the proposed one. This is because given others' proposed prices their utility is zero and they cannot obtain a positive utility. Relative to agent 1 the description of its selected price comes from maximizing its utility given the strategies selected by its rivals.

Now, to show that we have described all the equilibria, let us consider $\hat{P}$ that does not satisfy our assumptions. First, note that for $i \neq 1$, $\hat{p}_i \geq c_2$. This is because otherwise, if $\hat{p}_h = \min_{i \neq 1} \hat{p}_i < c_2$, then $\Psi_h^{\alpha R}\left(B_0, \hat{P}\right) > 0$; and thus $U_h\left(B_0, \hat{P}\right) < 0$. Second, $\hat{P}$ should also satisfy that $\hat{p}_h \geq c_h$ for each agent h such that $\Psi_h^{\alpha R}\left(B_0, \hat{P}\right) > 0$. Let us now assume that there is, at most, one agent i such that $\hat{p}_i = c_i = c_2$. Then agent i could improve its utility by selecting

$$\tilde{p}_i = \frac{\hat{p}_i + \min_{\{h:\hat{p}_h > \hat{p}_i\}} \hat{p}_h}{2} > c_2.$$

Therefore, given that, at least two agents other that 1 will select a price equals to c_2, the only possibility for agent 1 to play optimally comes from selecting the price that maximizes its utility guaranteeing a positive share. ∎

Finally, we describe the equilibria for the case in which at least three agents share the minimum cost. In such a case any equilibrium is such that no seller will obtain a positive utility.

Lemma A.3. 3. *Let assume that $c_1 = c_2 = c_3$. Then, any equilibrium $\bar{P}$ must satisfy* *that*

(a) $\bar{p}_i > c_1$ *for each* i *such that* $c_i > c_1$*;*

(b) $\bar{p}_i \geq c_1$ *for each* i *such that* $c_i = c_1$*; and*

(c) *There are at least three agents* i*,* j *and* h *such that* $\bar{p}_i = \bar{p}_j = \bar{p}_h = c_1$*.*

An argument similar to the one used to prove Lemma A.3.2 could be used to prove Lemma A.3.3.

APPENDIX 4. *A Proof of Theorem 6.6.*

This appendix is devoted to provide a formal proof of Theorem 6.6.

We will proceed as follows. First we will prove, Lemma A.4.1, that there exist α, and an equilibrium P^* for $\Gamma^{\alpha R}$ such that $S_0(B_0, P^*) > B_0 - c_2$. We then will see that the prices proposed by providers 1 and 2 are decreasing on the *relevant* budged constraint $\bar{B}$. As a conclusion of the above facts, and Proposition A.2.1, we reach the desired result.

Lemma A.4.1. *Let us assume that sellers' marginal costs are increasingly ordered, and* $0 \leq c_1 < c_2 < c_3 < B_0$*. Then there exist* $\alpha > 1$ *and* P^* *such that* P^* *is an equilibrium for* $\Gamma^{\alpha R}$*, and*

$$S_0(B_0, P^*) > B_0 - c_2.$$

Proof. Given B_0, c_2 and c_3, let us define the function $\tilde{B}: \mathbb{R} \to \mathbb{R}_+$ described by

$$\tilde{B}(\alpha) = \min\{B_0, (1+\alpha)c_3 - \alpha c_2\}$$

Now, for each $\alpha > 0$, let us define $P^*(a)$ by

$$\begin{aligned} p_i^*(\alpha) &= \tilde{B}(\alpha) \text{ for each } i > 2, \\ p_2^*(\alpha) &= \frac{\tilde{B}(\alpha) + \alpha c_2}{\alpha + 1}, \text{ and} \end{aligned}$$

$p_1^*(\alpha)$ is the unique solution on $\left(c_1, \tilde{B}(\alpha)\right)$ of the equation

$$\left(\tilde{B}(\alpha) - p_1^*(\alpha)\right) = \frac{1}{2}\left(\frac{\tilde{B}(\alpha) - p_2^*(\alpha)}{\tilde{B}(\alpha) - p_1^*(\alpha)}\right)^{\alpha}\left[\tilde{B}(\alpha) + (\alpha - 1)\, p_1^*(\alpha) - \alpha c_1\right] \tag{8.9}$$

Note that, by applying Lemma A.3.1, we know that for each $\alpha > 0$, $P^*(\alpha)$ is a Nash equilibrium for $\Gamma^{\alpha R}$. Now, by applying Lemma A.2.1, we have that there exist $\hat{\alpha} > 1$ such that

$$p_1^*(\hat{\alpha}) = \frac{\hat{\alpha}}{\hat{\alpha} - 1}c_2 - \frac{1}{\hat{\alpha} - 1}\tilde{B}(\hat{\alpha})$$

is a solution of Equation (8.9). Therefore, $P^*(\hat{\alpha})$ is an equilibrium for $\Gamma^{\hat{\alpha}R}$. Taking into account that

$$\tilde{B}(\hat{\alpha}) - p_1^*(\hat{\alpha}) = \tfrac{\hat{\alpha}}{\hat{\alpha}-1}\left(\tilde{B}(\hat{\alpha}) - c_2\right), \text{ and}$$

$$\tilde{B}(\hat{\alpha}) - p_2^*(\hat{\alpha}) = \tfrac{\hat{\alpha}}{\hat{\alpha}+1}\left(\tilde{B}(\hat{\alpha}) - c_2\right),$$

we have that

$$\frac{\tilde{B}(\hat{\alpha}) - p_2^*(\hat{\alpha})}{\tilde{B}(\hat{\alpha}) - p_1^*(\hat{\alpha})} = \left(\frac{\frac{\hat{\alpha}}{\hat{\alpha}+1}}{\frac{\hat{\alpha}}{\hat{\alpha}-1}}\right) = \frac{\hat{\alpha} - 1}{\hat{\alpha} + 1} < 1,$$

Therefore,

$$S_0\left(\tilde{B}(\hat{\alpha}), P^*(\hat{\alpha})\right) =$$
$$= \left[1 - \frac{1}{2}\left(\frac{\tilde{B}(\hat{\alpha}) - p_2^*(\hat{\alpha})}{\tilde{B}(\hat{\alpha}) - p_1^*(\hat{\alpha})}\right)^{\hat{\alpha}}\right]\left(\tilde{B}(\hat{\alpha}) - p_1^*(\hat{\alpha})\right) + \frac{1}{2}\left(\frac{\tilde{B}(\hat{\alpha}) - p_2^*(\hat{\alpha})}{\tilde{B}(\hat{\alpha}) - p_1^*(\hat{\alpha})}\right)^{\hat{\alpha}}\left(\tilde{B}(\hat{\alpha}) - p_2^*(\hat{\alpha})\right) >$$
$$> \frac{1}{2}\left(\tilde{B}(\hat{\alpha}) - p_1^*(\hat{\alpha})\right) + \frac{1}{2}\left(\tilde{B}(\hat{\alpha}) - p_2^*(\hat{\alpha})\right) = \frac{\hat{\alpha}^2}{\hat{\alpha}^2 - 1}\left(\tilde{B}(\hat{\alpha}) - c_2\right) > \left(\tilde{B}(\hat{\alpha}) - c_2\right).$$

So,

$$\begin{aligned} S_0\left(B_0, P^*(\hat{\alpha})\right) &= B_0 - \tilde{B}(\hat{\alpha}) + S_0\left(\tilde{B}(\hat{\alpha}), P^*(\hat{\alpha})\right) > \\ &> \left(B_0 - \tilde{B}(\hat{\alpha})\right) + \left(\tilde{B}(\hat{\alpha}) - c_2\right) = B_0 - c_2 \end{aligned}$$

Which is the desired result. ■

Proposition A.4 1. *Let us consider the implicit function*

$$F(B,p) = 2(B-p)^{\alpha+1} - (B-d)^{\alpha}(B + (\alpha-1)p - \alpha c) = 0,$$

with $\alpha > 1$*, then* p *is increasing in* B *whenever* $B > d > p > c \geq 0$*.*

Proof. First, let us observe that

$$\left.\frac{\partial F(B,p)}{\partial B}\right|_{F(B,p)=0} = 2(\alpha+1)(B-p)^{\alpha} - (B-d)^{\alpha} - \alpha(B-d)^{\alpha-1}(B-p+\alpha(p-c))$$

Since $F(B,p)=0$, the equation above implies that

$$(B-d)\left.\frac{\partial F(B,p)}{\partial B}\right|_{F(B,p)=0} = 2(\alpha+1)(B-p)^{\alpha}(B-d) - (B-d)^{\alpha+1}$$

$$-2\alpha(B-p)(B-d)^{\alpha}+2\alpha(B-p)^{\alpha+1} = 2\alpha(B-p)(B-d)\left((B-p)^{\alpha-1}-(B-d)^{\alpha-1}\right)+$$

$$+(B-d)\left((B-p)^{\alpha}-(B-d)^{\alpha}\right)+\left((B-d)+2\alpha(B-p)\right)(B-p)^{\alpha}>0$$

Now, let us observe that

$$\left.\frac{\partial F(B,p)}{\partial p}\right|_{F(B,p)=0} = -(\alpha-1)(B-d)^{\alpha} - 2(\alpha+1)(B-p)^{\alpha} < 0.$$

Therefore,

$$\left.\frac{dp}{dB}\right|_{F(B,p)=0} = -\frac{\frac{\partial F(B,p)}{\partial B}}{\frac{\partial F(B,p)}{\partial p}} > 0.$$

■

Lemma A.4. 2. *Let us assume that sellers' marginal costs are increasingly ordered, and* $0 \le c_1 < c_2 < c_3 < B_0$. *Then there exist* $\alpha > 1$ *such that for each* P^* *being a Nash* *equilibrium for* $\Gamma^{\alpha R}$,

$$S_0\left(B_0, P^*\right) > B_0 - c_2.$$

Proof. Given B_0, c_2 and c_3, let $\hat{\alpha}$ the one computed in Lemma A.4.1. For $\hat{\alpha}$ given, let P^* be a Nash equilibrium for $\Gamma^{\hat{\alpha} R}$. Note that, by Lemma A.3.1, $p_1^* < p_2^* < p_i^*$ for each $i > 2$. Let denote

$$\bar{B} = \min\left\{B_0; \min_{i \in \mathcal{I} \setminus \{1,2\}} p_i^*\right\}.$$

Let us observe that $\bar{B} \leq \tilde{B}\left(\hat{\alpha}\right)$, and

$$p_2^* = \frac{\bar{B} + \hat{\alpha} c_2}{\hat{\alpha} + 1}$$

So,

$$p_2^* \leq p_2^*\left(\hat{\alpha}\right) = \frac{\tilde{B}\left(\hat{\alpha}\right) + \hat{\alpha} c_2}{\hat{\alpha} + 1}.$$

On the other hand, note that, by Proposition A.4.1, we have that

$$p_1^* \leq p_1^*\left(\hat{\alpha}\right) = \frac{\hat{\alpha} c_2 - \tilde{B}\left(\hat{\alpha}\right)}{\hat{\alpha} - 1}.$$

Therefore

$$S_0\left(B_0,P^*\right)-\left(B_0+\bar{B}\right)=S_0\left(\bar{B},P^*\right)=$$

$$=\left[1-\frac{1}{2}\left(\frac{\bar{B}-p_2^*}{\bar{B}-p_1^*}\right)^{\hat{\alpha}}\right]\left(\bar{B}-p_1^*\right)+\frac{1}{2}\left(\frac{\bar{B}-p_2^*}{\bar{B}-p_1^*}\right)^{\hat{\alpha}}\left(\bar{B}-p_2^*\right)=$$

$$=\left(\bar{B}-\tilde{B}\left(\hat{\alpha}\right)\right)+\left[1-\frac{1}{2}\left(\frac{\tilde{B}\left(\hat{\alpha}\right)-p_2^*}{\tilde{B}\left(\hat{\alpha}\right)-p_1^*}\right)^{\hat{\alpha}}\right]\left(\tilde{B}\left(\hat{\alpha}\right)-p_1^*\right)+\frac{1}{2}\left(\frac{\tilde{B}\left(\hat{\alpha}\right)-p_2^*}{\tilde{B}\left(\hat{\alpha}\right)-p_1^*}\right)^{\hat{\alpha}}\left(\tilde{B}\left(\hat{\alpha}\right)-p_2^*\right)>$$

$$>\left(\bar{B}-\tilde{B}\left(\hat{\alpha}\right)\right)+\left[1-\frac{1}{2}\left(\frac{\tilde{B}\left(\hat{\alpha}\right)-p_2^*}{\tilde{B}\left(\hat{\alpha}\right)-p_1^*}\right)^{\hat{\alpha}}\right]\left(\tilde{B}\left(\hat{\alpha}\right)-p_1^*\left(\hat{\alpha}\right)\right)+\frac{1}{2}\left(\frac{\tilde{B}\left(\hat{\alpha}\right)-p_2^*}{\tilde{B}\left(\hat{\alpha}\right)-p_1^*}\right)^{\hat{\alpha}}\left(\tilde{B}\left(\hat{\alpha}\right)-p_2^*\left(\hat{\alpha}\right)\right)>$$

$$>\left(\bar{B}-\tilde{B}\left(\hat{\alpha}\right)\right)+\frac{1}{2}\left(\tilde{B}\left(\hat{\alpha}\right)-p_1^*\left(\hat{\alpha}\right)\right)+\frac{1}{2}\left(\tilde{B}\left(\hat{\alpha}\right)-p_2^*\left(\hat{\alpha}\right)\right)=\frac{\hat{\alpha}^2}{\hat{\alpha}^2-1}\left(\tilde{B}\left(\hat{\alpha}\right)-c_2\right)>$$

$$>\left(\bar{B}-\tilde{B}\left(\hat{\alpha}\right)\right)+\left(\tilde{B}\left(\hat{\alpha}\right)-c_2\right)=\bar{B}-c_2.$$

■